The 1:3:6 resonance

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Introduction

We investigate a three degree of freedom system with a phase space $\mathcal{P} \subset \mathbb{R}^6$. With $(q_1, q_2, q_3, p_1, p_2, p_3) \in \mathcal{P}$ and a Hamiltonian with a equilibrium point at the origin for which the Taylor expansion reads $H(q,p) = H_0^0(q,p) + H_1^0(q,p) + ...$ with $H_0^0 = \frac{q_1^2+p_1^2}{2} + 3\frac{q_2^2+p_2^2}{2} + 6\frac{q_3^2+p_3^2}{2}$ which describes a system of three harmonic oscillators which are in $1 : 3 : 6$ resonance. We will first work out the the normal form of the system and after that investigate the dynamics. Finally we will work out an example. The contents of this report will be based strongly on the lectures by Heinz Hanßman and his article on the $1 : 2 : 4$ resonance[2] with some support from the *Foundations of Mechanics* book[1].

Normal form theory

Firstly we want to determine the flow of H_0^0. As it is composed of three independent harmonic oscillators we have for the equations of motion.

$$\dot{q}_i = \frac{\partial H_0^0}{\partial p_i} = \omega_i p_i$$
$$\dot{p}_i = \frac{\partial H_0^0}{\partial q_i} = -\omega_i q_i$$

For $i \in \{1,2,3\}$ and where $\omega_1 = 1, \omega_2 = 3, \omega_3 = 6$. We then get the matrix

$$A = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 3 & & \\ & & -3 & 0 & & \\ & & & & 0 & 6 \\ & & & & -6 & 0 \end{pmatrix}$$

With the empty space being filled with zeros. Now the flow becomes $\phi_t^{H_0^0} = e^{tA}$. We can also simplify this by working in complex coordinates $z_i = \frac{q_i + ip_i}{\sqrt{2}}, \bar{z}_i =$

$\frac{q_i - ip_i}{\sqrt{2}}$. The equations of motion then become

$$\dot{z_i} = \frac{\dot{q_i} + i\dot{p_i}}{\sqrt{2}} = \frac{\omega_i p_i - i\omega_i q_i}{\sqrt{2}} = -i\omega_i z_i$$

Which is simply solvable to get $\phi_t^{H_0^0} = (e^{-it}z_1, e^{-3it}z_2, e^{-6it}z_3)$. Which is a free action, but for $z_1 = 0$ we get an $\mathbb{Z}_3$ isotropy and for $z_1 = z_2 = 0$ we get an $\mathbb{Z}_6$ isotropy.

Next we want to find the spectrum of the linear mapping $X_{H_0^0} : \mathcal{G}_{j+2} \longrightarrow \mathcal{G}_{j+2}$. For this we will work in complex coordinates so it is usefull to first rewrite the canonical poisson bracket into its complex version. We have the Wirtinger derivatives

$$\frac{\partial}{\partial q_i} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i}\right),\ \frac{\partial}{\partial p_i} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial \bar{z}_i}\right)$$

And putting these into the canonical poisson bracket

$$\{F, G\} = \sum_{i=1} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

We get

$$\{F, G\} = -i\sum_{i=1} \frac{\partial F}{\partial z_i}\frac{\partial G}{\partial \bar{z}_i} - \frac{\partial F}{\partial \bar{z}_i}\frac{\partial G}{\partial z_i}$$

We have then in complex coordinates $H_0^0 = z_1\bar{z}_1 + 3z_2\bar{z}_2 + 6z_3\bar{z}_3$ and the individual brackets become

$$\begin{aligned}\{z_i, H_0^0\} &= -i(\omega_i z_i)\\ \{\bar{z}_i, H_0^0\} &= i(\omega_i \bar{z}_i)\end{aligned}$$

This is much alike to the exercises and so we do the same thing as we did there. We get for any monomial $z_1^{\alpha_1}\bar{z}_1^{\beta_1}z_2^{\alpha_2}\bar{z}_2^{\beta_2}z_3^{\alpha_3}\bar{z}_3^{\beta_3}$ the following

$$\begin{aligned}\{z_1^{\alpha_1}\bar{z}_1^{\beta_1}z_2^{\alpha_2}\bar{z}_2^{\beta_2}z_3^{\alpha_3}\bar{z}_3^{\beta_3}, H_0^0\} &= \alpha_1\{z_1, H_0^0\}z_1^{\alpha_1-1}\bar{z}_1^{\beta_1}z_2^{\alpha_2}\bar{z}_2^{\beta_2}z_3^{\alpha_3}\bar{z}_3^{\beta_3} + \ldots\\ &\quad + \beta_1\{\bar{z}_3, H_0^0\}z_1^{\alpha_1}\bar{z}_1^{\beta_1}z_2^{\alpha_2}\bar{z}_2^{\beta_2}z_3^{\alpha_3}\bar{z}_3^{\beta_3-1}\\ &= i\left(\omega_1(\beta_1 - \alpha_1) + \omega_2(\beta_2 - \alpha_2) + \omega_3(\beta_3 - \alpha_3)\right)z_1^{\alpha_1}\bar{z}_1^{\beta_1}z_2^{\alpha_2}\bar{z}_2^{\beta_2}z_3^{\alpha_3}\bar{z}_3^{\beta_3}\end{aligned}$$

Meaning every monomial is a complex eigenvector. Now because $\mathcal{G}_{j+2}$ consists of linear combinations of monomials and the Poisson bracket distributes linearly we have found the entire spectrum of $X_{H_0^0}$ acting on $\mathcal{G}_{j+2}$.

We now want to determine ker $X_{H_0^0}$ and im $X_{H_0^0}$ and split $\mathcal{G}_{j+2}$ accordingly. This follows the same program and formal context as exercise 24. Doing this for $j \in \{1, 2\}$ we find that at least any monomial with $\alpha_i = \beta_i$ for all i should be in the kernel. For degree 2 these are exactly the $\tau_i = z_i\bar{z}_i$. But also on all other even degrees such as degree 4 we get combinations of these such as

$$\tau_1^2, \quad \tau_2^2, \quad \tau_3^2, \quad \tau_1\tau_2, \quad \tau_2\tau_3, \quad \tau_1\tau_3$$

These are all combinations as any other monomial with this property would be of degree 5 or higher. Given then $\alpha_i \neq \beta_i$ and $\sum_{i=1}^{3} \alpha_i + \beta_i \leq 4$ we also get as monomials in the kernel

$$z_2^2\bar{z}_3, \quad \bar{z}_2^2 z_3, \quad z_1^3\bar{z}_2, \quad \bar{z}_1^3 z_2$$

We call these in order $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Completing the basis of the kernel. Now all other monomials must necessarily have non-zero eigenvalue $\lambda \neq 0$ and are in the image by

$$z_1^{\alpha_1}\bar{z}_1^{\beta_1}z_2^{\alpha_2}\bar{z}_2^{\beta_2}z_3^{\alpha_3}\bar{z}_3^{\beta_3} = \{\frac{1}{\lambda}z_1^{\alpha_1}\bar{z}_1^{\beta_1}z_2^{\alpha_2}\bar{z}_2^{\beta_2}z_3^{\alpha_3}\bar{z}_3^{\beta_3}, H_0^0\}$$

Achieving the splitting of $\mathcal{G}_{j+2}$ for $j \in \{1, 2\}$. If we investigate $j = 3$ we find two different new types of monomials which enter into the kernel, namely a set of cross-terms

$$\sigma_1\tau_1, \quad \sigma_1\tau_2, \quad \sigma_1\tau_3, \quad \sigma_2\tau_1, \quad \sigma_2\tau_2, \quad \sigma_2\tau_3$$

and a set of new terms

$$z_1^3 z_2\bar{z}_3, \quad \bar{z}_1^3\bar{z}_2 z_3$$

Which are gained from solving the same equations as before for $j = 3$.

Converting these back to real monomials we again use $z_i = \frac{q_i + ip_i}{\sqrt{2}}, \bar{z}_i = \frac{q_i - ip_i}{\sqrt{2}}$ and $p_i = \frac{z_i - \bar{z}_i}{i\sqrt{2}}$. Substituting these into the monomials in the kernel we find first the constituent parts of H_0^0

$$\frac{q_1^2 + p_1^2}{2}, \quad \frac{q_2^2 + p_2^2}{2}, \quad \frac{q_3^2 + p_3^2}{2}$$

But then terms such as $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are not real and therefore cannot be part of a real basis. But because these terms appear as conjugate pairs we can still get pairs of real bases from them using $\mathrm{Re}(\sigma_1) = \frac{z_1 + \bar{z}_1}{\sqrt{2}}$ and $\mathrm{Im}(\sigma_1) = \frac{z_1 - \bar{z}_1}{i\sqrt{2}}$. Doing this for σ_1 and σ_2 we get

$$q_2^2 q_3 - q_3 p_2^2 + 2q_2 p_2 p_3, \quad 2q_2 q_3 p_2 - q_2^2 p_3 + p_2^2 p_3$$

And a similar thing can be done for higher order terms.
We could then follow the program laid out here in order to get higher order normal forms of the Hamiltonian. We should expect combinations of the τ_i and σ_i already found at higher degrees and new terms $\sigma_7 = z_1^6\bar{z}_3$ and its conjugate σ_8 at degree 7. Going higher from that we should expect all other terms which appear to be cross-terms of what we have already found.

Dynamics

Now we seek to answer if the cubic normal form can be a non-degenerate integrable system. The general truncated normal form of order three with detuning parameters λ_1 and λ_2 looks like

$$H_3 = \tau_1 + (1+\lambda_1)\tau_2 + (1+\lambda_2)\tau_3 + a\mathrm{Re}(\sigma_1) + b\mathrm{Im}(\sigma_1)$$

with a and b constants. We can rotate our coordinate system to get $b=0$ and rescale to get $a=1$. Which gets us

$$H_3 = \tau_1 + (1+\lambda_1)\tau_2 + (1+\lambda)\tau_3 + \mathrm{Re}(\sigma_1)$$

Now because $\tau_1 = z_1\bar{z}_1$ is still uncoupled we have that the system is degenerate. For integrability in three degrees of freedom we need there to be three conserved quantities under the flow of this Hamiltonian. We already have two conserved quantities H_3 and H_0^0. We can also observe that σ_1 does not involve z_1 or $\bar{z}_1$ therefore we get $\{\tau_1, H_3\} = 0$ making τ_1 into our third conserved quantity giving us the desired integrability.

We now have that we have a free action S^1 from H_0^0 as we normally would, but we also have a free action S^1 from the independence of τ_1. In this case we get thus two level sets and a reduction by $S^1 \times S^1 = \mathbb{T}$.

$$\mathcal{P}_{\eta_1,\eta_2} = H_0^{0\;-1}(\eta_1) \cap \tau_1^{-1}(\eta_2)/\mathbb{T}^2$$

with $(\tau_2, \mathrm{Re}(\sigma_1), \mathrm{Im}(\sigma_1)) \in \mathbb{R}^3$ and $\tau_2 \geq 0$. For this space we also have the syzygy relations $\sigma_1^2 + \sigma_2^2 = \tau_2^2\tau_3$. Which gets us close to the $1:2$ resonance picture we talked about in the lecture. The main difference being that now the isotropies are different. Namely if we set $\tau_1 = \eta_2 = 0$, we get a $\mathbb{Z}_3$ group and if we then also set $\tau_2 = 0$ we would get a $\mathbb{Z}_6$ group.

If we then move on to the general quartic normal form. We expect the following terms

$$\tau_1, \tau_2, \tau_3, \tau_1^2, \tau_2^2, \tau_3^2, \tau_1\tau_2, \tau_2\tau_3, \tau_1\tau_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4$$

where σ_3, σ_4 are as described before

$$z_1^3\bar{z}_2. \quad \bar{z}_1^3 z_2$$

The general quartic normal form with detuning then becomes

$$H_4 = \tau_1 + 3\tau_2 + 6\tau + \alpha_1\tau_1^2 + \alpha_2\tau_2^2 + \alpha_3\tau_3^2 + \alpha_4\tau_1\tau_2 + \alpha_5\tau_2\tau_3 + \alpha_6\tau_1\tau_3 + a\sigma_1 + \sigma_3$$

Where we have rotated out σ_2 and σ_4. The coefficients α_i are then determined by the cubic potential being normalised. In the general case where $\alpha_i \neq 0$ for all i we have that then the Hamiltonian system is not integrable because $\{\tau_1, \sigma_3\} \neq 0$ making it so that τ_1 is no longer a conserved quantity as before.

We then find that the isotropy points remain unchanged with a $\mathbb{Z}_3$ isotropy at $\tau_1 = 0$ and a $\mathbb{Z}_6$ isotropy if τ_2 is zero as well. We also have a set of syzygies $\sigma_1^2 + \sigma_2^2 = \tau_2^2\tau_1$ and $\sigma_3^2 + \sigma_4^2 = \tau_1^3\tau_2$, which enforce some structure on the reduced phase space. Because then further the reduced phase space is still a 4 dimensional space it becomes difficult to say more general things about it.

Worked example

We will be working out the cubic potential $H_1^0 = \frac{q_2}{2}(q_1^2 + q_2 q_3)$. We use first substitution into complex variables to get

$$\begin{aligned} H_1^0 &= \frac{q_2}{2}(q_1^2 + q_2 q_3) \\ &= \frac{z_2 + \bar{z}_2}{\sqrt{2}^3}\left(\left(\frac{z_1 + \bar{z}_1}{\sqrt{2}}\right)^2 + \frac{z_2 + \bar{z}_2}{\sqrt{2}}\frac{z_3 + \bar{z}_3}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}^5}((z_1 + \bar{z}_1)^2(z_2 + \bar{z}_2) + (z_2 + \bar{z}_2)^2(z_3 + \bar{z}_3)) \\ &= \frac{1}{\sqrt{2}^5}(z_1^2 z_2 + 2z_1\bar{z}_1 z_2 + \bar{z}_1^2 z_2 + z_1^2\bar{z}_2 + 2z_1\bar{z}_1\bar{z}_2 + \bar{z}_1^2\bar{z}_2 \\ &\qquad + z_2^2 z_3 + 2z_2\bar{z}_2 z_3 + \bar{z}_2^2 z_3 + z_2^2\bar{z}_3 + 2z_2\bar{z}_2\bar{z}_3 + \bar{z}_2^2\bar{z}_3) \end{aligned}$$

Now, after normalization only the resonant monomials survive getting us

$$\begin{aligned} H_1^1 &= (\sqrt{2})^5(\bar{z}_2^2 z_3 + z_2^2\bar{z}_3) \\ &= q_2^2 q_3 - q_3 p_2^2 - 2q_2 p_2 p_3 \end{aligned}$$

Which may be recognised as σ_1 and σ_2. As our integrability and degeneracy arguments argued a general case they are also applicable here, making it so that this is a degenerate, integrable system.

The normalization procedure for the quartic normal form has too many parts to do by hand, but what we are interested in is mostly the fact if any σ_3 or σ_4 terms result from this process and we have that the coefficients of σ_3 and σ_4 must be zero, which is a surprising result. It follows from the fact that we require terms for which the Poisson bracket equals $z_1^3\bar{z}_2$, but the only terms in H_3 with z_1^2 are $z_1^2 z_2$ and $z_1^2\bar{z}_2$, but there exists no term $z_1\bar{z}_2^2$ or $z_1 z_2^2$ in H_3. Therefore it is algebraically impossible to get a σ_3 or σ_4 term in your fourth order hamiltonian. This means that again H_4 becomes integrable.

How about non-degeneracy then? Well for that we need some coupling term between z_1 and z_2 or z_3. But we have that the coefficients of σ_3 and σ_4 are zero, then it must be an action term which provides this coupling. So for non-degeneracy we want either $\tau_1\tau_2 \neq 0$ or $\tau_1\tau_3 \neq 0$. Now a $\tau_1\tau_2$ term is actually easily found as we have

$$\tau_1\tau_2 = \{z_1^2 z_2, \bar{z}_1^2\bar{z}_2\} = z_1\bar{z}_1 z_2\bar{z}_2$$

Both of which are found in H_3. Getting us non-degeneracy.

Now as we normalize further the question becomes if integrability keeps being true for H_5. This depends on the appearance of σ_5, σ_6 or $\sigma_3\tau_i$ terms for all i. Although it is speculative my intuition points to these terms making it into the normal form. As the terms are of degree 5 we are looking for some combination of a degree 3 term, for which we only have the monomials σ_1, σ_2 and a degree 4 term, which, because we haven't yet normalised consists of resonant and non-resonant terms.

References

[1] Ralph Abraham and Jerrold E. Marsden. *Foundations of Mechanics.* AMS Chelsea Publishing/American Mathematical Society, 1978. ISBN: 9780821844380.

[2] Heinz Hanßmann, Reza Mazrooei-Sebdani, and Ferdinand Verhulst. "The 1:2:4 Resonance in a Particle Chain". In: *Indagationes Mathematicae* 32 (June 2020). DOI: 10.1016/j.indag.2020.06.003.