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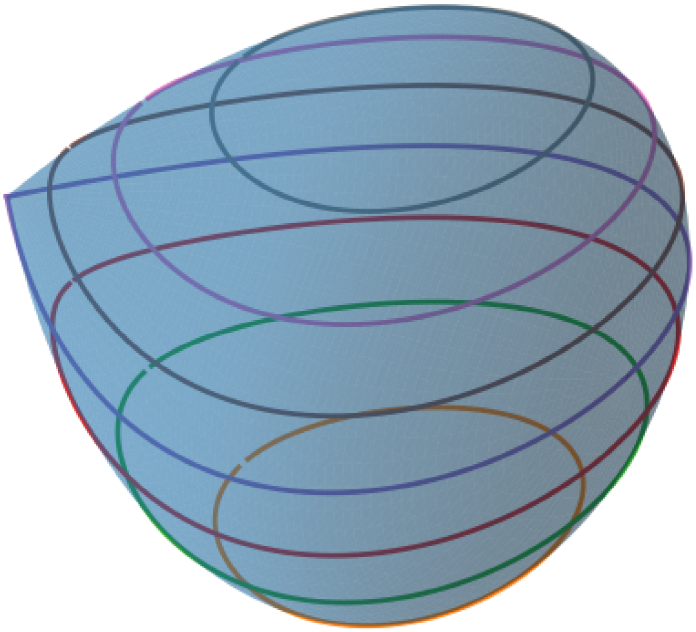
Normal forms and dynamics of the 2:3:6 resonance

Final Project Geometric Mechanics

Chris Vijgenboom

Student number: 0292060. E-mail: c.vijgenboom@students.uu.nl

Mathematical Sciences



Supervisor:

Dr. H. Hanßmann
Mathematical Institute

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1 Introduction

When studying elliptic equilibria in Hamiltonian systems, a useful starting point is considering only the quadratic part of the Hamiltonian. The dynamics defined by this quadratic part boils down to the dynamics of uncoupled harmonic oscillators. These harmonic oscillators have certain frequencies/periods and alignment of these frequencies can cause big changes in the dynamics of the complete system. For example, consider some Hamiltonian of which the quadratic part defines two decoupled harmonic oscillators. The frequency of one of the oscillators is precisely twice as big as the frequency of the other oscillator. Then the higher order terms in the Hamiltonian have a larger effect on the full dynamics than if those oscillators would not have been in resonance. This phenomenon is a large part of what is being studied in normal form theory.

In this report we study a specific case of resonance, namely the 2:3:6 resonance. Before we start with this we first give a small review of the necessary Hamiltonian and normal form theory in chapter 2. Then in chapter 3 we apply this to the 2:3:6 resonance by examining the cubic and quartic normal form. The cubic normal form will turn out to define a completely integrable system. This means that the dynamics are much simpler to study. For the quartic normal form this is not necessarily the case. However, we apply our findings for the cubic and quartic normal form to a specific Hamiltonian. The quartic normal form of this specific Hamiltonian will turn out to define a completely integrable system. At the end of this chapter we give some remarks on what can happen if we normalize further.

2 Hamiltonian systems and normal forms

In this section we quickly review some of the most important notions that arise in Hamiltonian systems and normal forms. For this small section I have taken inspiration from chapter 10 in [SVM07]. Consider a manifold $\mathcal{M}$ and its cotangent bundle $T^*\mathcal{M}$. With $\mathcal{M}$ we associate the configuration space and with $T^*\mathcal{M}$ the phase space. The dimension of $\mathcal{M}$ will be referred to as the number of degrees of freedom. On the cotangent bundle we can define a symplectic form ω such that every function $H : T^*M \to \mathbb{R}$ defines a *Hamiltonian vector field* X_H on the phase space via the following relation:

$$\iota_{X_H}\omega = dH.$$

In canonical local coordinates $(x, y) \in T^*\mathcal{M}$ this reduces to the equations of motion:

$$\begin{cases} \dot{x} &= \frac{\partial}{\partial y}H \\ \dot{y} &= -\frac{\partial}{\partial x}H. \end{cases}$$

On the phase space we can define for any two smooth $F, G : T^*\mathcal{M} \to \mathbb{R}$ the Poisson bracket $\{F, G\} = \omega(X_F, X_G)$. In our case of a Hamiltonian H we get that for any smooth function F we have that

$$\dot{F} = X_H(F) = \{F, H\}.$$

A particularly important class of Hamiltonian systems are *completely integrable Hamiltonian systems*. These are systems that include n degrees of freedom that have n conserved quantities or integrals $F_1, ..., F_n$. These integrals must be functionally independent and in involution, i.e. $\{F_i, F_j\} = 0,\ i, j = 1, ..., n$. These systems are particularly interesting since the dynamics can be easily studied. In fact, the Liouville-Arnold theorem tells us that for completely integrable systems we have *action-angle coordinates* (I, ϕ) which simplify the dynamics to quasi-periodic motion on invariant tori. In these coordinates the Hamiltonian can be written in terms of only the action $H(I)$ which yield the equations of motion

$$\begin{cases} \dot{\phi} &= \frac{\partial}{\partial I}H(I) \\ \dot{I} &= 0 \end{cases}.$$

The map $\alpha(y) = \frac{\partial}{\partial I}H(I)$ is known as the frequency map. If the Jacobian with respect to the actions of this map is invertible, the whole system is said to be *non-degenerate*. This condition ensures that almost all tori have dense orbits [Han06]. A natural question is to ask if these integrable systems retain these motions once a small perturbation is introduced to the system. This leads to the KAM-theorem which roughly states that most of the invariant tori survive if a completely integrable non-degenerate Hamiltonian system undergoes a small perturbation. Since these systems are so well-understood, the question arises if we can expand these ideas to general Hamiltonian systems.

Now consider a general Hamiltonian system at equilibrium at the origin. There we can expand the Hamiltonian like

$$H = \sum_{k=0}^{\infty} \frac{H_k^0}{k!}.$$

where H_k^0 is in the space $\mathcal{G}_{j+2}$ of homogeneous real polynomials of degree $j + 2$. This system is in general not an integrable system. However, around this equilibrium the quadratic part of the Hamiltonian governs most of the dynamics and we can in some sense view all higher order terms as a perturbation. Furthermore, around an elliptic equilibrium there exists some symplectic change of variables such that H_0^0 can always be written in the form without mixed terms, so for n degrees of freedom:

$$H_0^0 = \sum_{i=1}^{n} \omega_i \frac{x_i^2 + y_i^2}{2}.$$

We call $\omega = (\omega_1, ..., \omega_n)$ the frequency vector. We see that the quadratic part of the Hamiltonian reduces to n decoupled harmonic oscillators. The Hamiltonian system defined by H_0^0 is a simple system that is understood well. It is in fact an integrable system and the details of this system will be discussed later. Most importantly, the dynamics are quasi-periodic orbits on invariant tori. If H_0^0 even defines a non-degenerate system, then we can apply the KAM-theorem to the full system to conclude that around this equilibrium most of these invariant tori survive in the general Hamiltonian system. Unfortunately H_0^0 defines a degenerate integrable system since the frequencies are constant and hence the Hessian is the zero matrix with determinant zero. So we cannot apply the KAM-theorem. This idea does however signify the main approach that will be used when formulating normal forms.

Normal forms are new Hamiltonians which are derived from the original Hamiltonian by performing a symplectic change of variables m times. The first change of variables has the purpose of removing as many third order terms as possible from H. The second change of variables wants to remove as many fourth order terms as possible. This process continues. The hope is to end up with a Hamiltonian that, up to the polynomials of order m, defines a completely integrable non-degenerate system. Then all higher order terms can be seen as a perturbation and we can apply the KAM-theorem. To define such a symplectic change of variables, we choose some Hamiltonian W and compose H with the time-1 flow of the vector field defined by W. This is a symplectic map. The optimal choice of W depends on H and can be chosen accordingly. To illustrate this, the first step in the normalization process will go as follows:

$$\begin{aligned}\mathcal{H} = H \circ \varphi_{t=1}^W &= H + \{H, W\} + \frac{1}{2}\{\{H, W\}, W\} + \text{h.o.t.}\\ &= H_0^0 + H_1^0 + \{H_0^0 + H_1^0, W\} + \frac{1}{2}\{\{H_0^0 + H_1^0, W\}, W\} + \text{h.o.t.}\\ &= H_0^0 + H_1^0 + \{H_0^0, W\} + \{H_1^0, W\} + \frac{1}{2}\{\{H_0^0, W\}, W\} + \frac{1}{2}\{\{H_1^0, W\}, W\} + \text{h.o.t..}\end{aligned}$$

Then the cubic normal form $\mathcal{H}_3$ is the same as $\mathcal{H}$ but truncated after the third order polynomials. The resulting third order normal form will be

$$\mathcal{H}_3 = H_0^0 + H_1^0 + \{H_0^0, W\}.$$

Ideally we choose W in such a way that $H_1^0 = \{W, H_0^0\}$ to eliminate all third order terms. This would be possible if all functions in H_1^0 are in the image of the linear mapping $\{\cdot, H_0^0\} : \mathcal{G}_{j+2} \to \mathcal{G}_{j+2}$. Therefore to determine if such a W exists we must examine the linear mapping $\{\cdot, H_0^0\} : \mathcal{G}_{j+2} \to \mathcal{G}_{j+2}$. Note that by the definition of the Poisson bracket this linear mapping can be viewed as the vector field $X_{H_0^0}$. To investigate this linear mapping it is useful to first switch to complex variables on the space $\mathcal{G}_{j+2}^{\mathbb{C}}$ of homogeneous complex polynomials of order $j+2$. Complex variables make all computations easier and after these computations it is possible to switch back to real variables by taking the real and imaginary parts of variables. This space admits the splitting

$$\mathcal{G}_{j+2}^{\mathbb{C}} = \ker(X_{H_0^0}) \oplus \text{im}(X_{H_0^0}).$$

This splitting makes apparent that we cannot find a W that eliminates all complex polynomials if the kernel is non-empty. The kernel is spanned by all eigenvectors of the mapping $X_{H_0^0}$ with eigenvalue 0. If the ratios between the components of the frequency vector ω are irrational, this kernel consists only of polynomials that can be written in terms of $\frac{x_i^2+y_i^2}{2}$, $i = 1, ..., n$. If however the ratios are rational, for simplicity say $\omega \in \mathbb{Z}^n$, then the kernel can become much larger. All monomials in $\ker(X_{H_0^0})$ we will call *resonant monomials of order* $j+2$. If there are no third order resonant monomials, the cubic normal form becomes just the quadratic Hamiltonian and hence we have simplified the system drastically. Even if the kernel is non-empty, it is much smaller than the image and we end up with a simplified Hamiltonian in any case. The hope is to conclude this simplified Hamiltonian defines a non-degenerate integrable system. If this is not the case, we can normalize again using the same procedure with some Hamiltonian W' of fourth order to obtain a quartic normal form $\mathcal{H}_4$. In particular we find that the non-resonant terms can be removed with some symplectic change of variables but the resonant terms remain. These resonant terms then really determine the dynamics of the normal forms and that is what will be studied in the next section for the 2:3:6 resonance.

3 Normal forms of the 2:3:6 resonance and its dynamics

In this section we study the dynamics of a Hamiltonian system defined by the Hamiltonian $H(x,y) = H_0^0(x,y) + H_1^0(x,y) + \dots$ around an elliptic equilibrium at the origin with quadratic part

$$H_0^0(x,y) = 2\frac{y_1^2 + x_1^2}{2} + 3\frac{y_2^2 + x_2^2}{2} + 6\frac{y_3^2 + x_3^2}{2} \quad .$$

The corresponding frequency vector is $\omega = (\omega_1, \omega_2, \omega_3) := (2, 3, 6)$.
To investigate this system we recall some identities we have computed in past exercises. To avoid overcomplicating the notation, if i is written in subscript it is an index, if i is not written in subscript it denotes $\sqrt{-1}$. Define complex variables

$$z_i = x_i + iy_i, \quad i = 1, 2, 3.$$

The complex variables satisfy the following Poisson brackets

$$\{z_i, z_j\} = 0, \quad i, j = 1, 2, 3 \quad ,$$

and

$$\{z_i, \bar{z}_j\} = -2i\delta_{ij}, \quad i, j = 1, 2, 3 \quad .$$

These identities give rise to the basic invariants

$$\tau_i = \frac{z_i \bar{z}_i}{2}, \quad i = 1, 2, 3 \quad .$$

These basic invariants satisfy the Poisson brackets

$$\{\tau_i, \tau_j\} = 0, \quad i, j = 1, 2, 3 \quad .$$

Then the quadratic Hamiltonian becomes

$$H_0^0 = \omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3 \quad .$$

Furthermore we have the Poisson brackets

$$\{z_i, H_0^0\} = -2i\omega_i z_i \text{ and } \{\bar{z}_i, H_0^0\} = 2i\omega_i \bar{z}_i, \quad i = 1, 2, 3 \quad .$$

By repeated application of Leibniz' rule we also get for any $n \in \mathbb{N}$

$$\{z^n, H_0^0\} = nz^{n-1}\{z, H_0^0\}.$$

As we have seen in the previous section, to study the normal forms we must first study properties of the linear mapping $X_{H_0^0}$.

Since we are talking about a Hamiltonian system the equations of motion become

$$\begin{cases} \dot{x}_i &= \frac{\partial}{\partial y_i} H_0^0 = \omega_i y_i \\ \dot{y}_i &= -\frac{\partial}{\partial x_i} H_0^0 = -\omega_i x_i. \end{cases}$$

for $i = 1, 2, 3,$. This is a harmonic oscillator with general solution

$$\begin{cases} x_i(t) &= x_i(0)\cos(\omega_i t) + y_i(0)\sin(\omega_i t) \\ y_i(t) &= y_i(0)\cos(\omega_i t) - x_i(0)\sin(\omega_i t). \end{cases}$$

Then the flow is given by the general solution for any initial condition. For $x, y \in \mathbb{R}^3$:

$$\varphi_t(x,y) = (x\cos(\omega t) + y\sin(\omega t), y\cos(\omega t) - x\sin(\omega t)) \,,$$

or in complex coordinates

$$\varphi_t(z_1, z_2, z_3) = (e^{-2it} z_1, e^{-3it} z_2, e^{-6it} z_3).$$

We now investigate the spectrum of the linear mapping $X_{H_0^0}$ which will help us find the resonant monomials. The resonant monomials are the monomials that are invariant under the flow defined by H_0^0 and are given by the eigenvectors with eigenvalue 0 of the linear mapping $X_{H_0^0}$. Recall that for any smooth function F we have that $X_{H_0^0}(F) = \{F, H_0^0\}$ and define the complex monomial of degree $j+2$ as

$$M_{a,b,c,d,e,f} = z_1^a \bar{z}_1^b z_2^c \bar{z}_2^d z_3^e \bar{z}_3^f \text{ with } a+b+c+d+e+f = j+2.$$

These form a basis of the space $\mathcal{G}_{j+2}^{\mathbb{C}}$ of complex monomials of degree $j+2$. This yields the relation

$$\begin{aligned}
\{M_{a,b,c,d,e,f}, H_0^0\} &= \Bigg(\frac{a}{z_1}\{z_1, H_0^0\} + \frac{b}{\bar{z}_1}\{\bar{z}_1, H_0^0\} + \frac{c}{z_2}\{z_2, H_0^0\} + \frac{d}{\bar{z}_2}\{\bar{z}_2, H_0^0\} \\
&\quad + \frac{e}{z_3}\{z_3, H_0^0\} + \frac{f}{\bar{z}_3}\{\bar{z}_3, H_0^0\} \Bigg) M_{a,b,c,d,e,f} \\
&= \left(\frac{a}{z_1}(-4iz_1) + \frac{b}{\bar{z}_1} 4i\bar{z}_1 + \frac{c}{z_2}(-6iz_2) + \frac{d}{\bar{z}_2} 6i\bar{z}_2 + \frac{e}{z_3}(-12iz_3) + \frac{f}{\bar{z}_3} 12i\bar{z}_3 \right) M_{a,b,c,d,e,f} \\
&= 2i\left(2(b-a) + 3(d-c) + 6(f-e)\right) M_{a,b,c,d,e,f} \quad .
\end{aligned}$$

The first equality follows from a similar calculation as we have done in exercise 28. We conclude that $M_{a,b,c,d,e,f}$ form a basis of eigenvectors with eigenvalues $2i\left(2(b-a) + 3(d-c) + 6(f-e)\right)$. To determine which monomials form a basis for each part of the splitting we need to check which monomials have eigenvalue 0, i.e. which combinations of a, b, c, d, e, f satisfy both relations $2(b-a) + 3(d-c) + 6(f-e) = 0$ and $a+b+c+d+e+f = j+2$. This gives the following options up to at least order 5:

i) $(b-a) = (d-c) = (f-e) = 0$

ii) $(b-a) = \mp 3,\ (d-c) = 0,\ (f-e) = \pm 1$

iii) $(b-a) = 0,\ (d-c) = \mp 2,\ (f-e) = \pm 1$

iv) $(b-a) = \mp 3,\ (d-c) = \pm 2,\ (f-e) = 0$

v) $(b-a) = \mp 6,\ (d-c) = \pm 3,\ (f-e) = 0$

vi) $(b-a) = \mp 6,\ (d-c) = 0,\ (f-e) = \pm 2$

vii) $(b-a) = 0,\ (d-c) = \mp 4,\ (f-e) = \pm 2$

Note that for each option, if we sum the absolute values of the numbers in that option, the lowest order possible monomial is of that order. Then any other possible monomial in this option must have order some multiple of 2 added to this lower order. For example, in option iii), the lowest order possible monomial is order $2+1=3$. All monomials in this option must be of order $3+2n,\ n \in \mathbb{N}_0$. To see this, note that if we want to create another monomial satisfying this option, we for example need to add 1 to a. To keep satisfying the condition we must in turn add 1 to b. This increases the order of the monomial by 2. In particular, all monomials are of odd order in this example.
For $j = 1, 2, 3$ we can immediately discard options v),vi),vii) and all other possibilities since they would require a monomial of higher order. Now we treat each j by cases:

- *j=1:* Option i) gives no possibilities since this would mean that a and b are both even or odd. The same can be said for the combination of c and d and the combination of e and f. To add two even numbers gives another even number. To add two odd numbers gives an even number. Hence $a+b+c+d+e+f$ can never add up to 3 which is odd. Option ii) and iv) are immediately impossible since that would require a monomial which is of higher order than 3. Option iii) gives rise to the resonant monomials which span the kernel of $X_{H_0^0}$

$$z_2^2\bar{z}_3, \quad \bar{z}_2^2 z_3.$$

All remaining monomials span the image of $X_{H_0^0}$.

- *j=2:* Option i) gives rise to the monomials

  $$z_i\bar{z}_i z_{i'}\bar{z}_{i'}, \ i, i' = 1, 2, 3.$$

  These can all be written in terms of the basic invariants τ_1, τ_2, τ_3. Option ii) yields the monomials

  $$z_1^3\bar{z}_3, \ \bar{z}_1^3 z_3.$$

  Option iii) yields no possibilities since the lowest order possible monomial is of order 3 hence each possible monomial has to be of odd order. The other options require a higher order monomial hence these monomials span the kernel. The remaining monomials span the image.

- *j=3:* Option i) and ii) are not possible because all monomials must be of odd order. Option iii) yields the monomials

  $$z_1\bar{z}_1\bar{z}_2^2 z_3, \ z_1\bar{z}_1 z_2^2\bar{z}_3, \ z_2\bar{z}_2^3 z_3, \ z_2^3\bar{z}_2\bar{z}_3, \ z_2^2 z_3\bar{z}_3^2, \ \bar{z}_2^2 z_3^2\bar{z}_3.$$

  Note that these monomials are the product of one second order resonant monomial ($\tau_1, \ \tau_2$ or τ_3) and one third order resonant monomial. Option iv) yields the monomials

  $$z_1^3\bar{z}_2^2, \ \bar{z}_1^3 z_2^2.$$

  These monomials span the kernel. The remaining monomials span the image.

We have computed the resonant monomials using complex variables z_i. However the Hamiltonian should be defined using real functions. Every monomial that can be written in terms of the invariants τ_1, τ_2, τ_3 are already real monomials. Then note that for each monomial $M_{a,b,c,d,e,f}$ in the kernel the monomial $\overline{M}_{a,b,c,d,e,f} = M_{b,a,d,c,f,e}$ is also in the kernel. Consider the case $j = 1$; to make the kernel real, define

$$\sigma_1 = \Re(z_2^2\bar{z}_3), \quad \sigma_2 = \Im(z_2^2\bar{z}_3).$$

These will function as the real basis vectors of the kernel and hence be the real resonant monomials of order 3. Apply the same strategy as for the kernel to the image to obtain a real basis for the image. For higher orders we define similarly

$$\sigma_3 = \Re(z_1^3\bar{z}_3), \quad \sigma_4 = \Im(z_1^3\bar{z}_3), \quad \sigma_5 = \Re(z_1^3\bar{z}_2^2), \quad \sigma_6 = \Im(z_1^3\bar{z}_2^2).$$

We do not need to define new invariants for the real and imaginary parts of the monomials

$$z_2\bar{z}_2^3 z_3, \ z_2^3\bar{z}_2\bar{z}_3, \ z_2^2 z_3\bar{z}_3^2, \ \bar{z}_2^2 z_3^2\bar{z}_3,$$

since the real and imaginary parts of these monomials can be written in terms of tau's and sigma's. We get

$$\begin{aligned}
\Re(z_1\bar{z}_1\bar{z}_2^2 z_3) &= 2\tau_1\sigma_1 \\
\Im(z_1\bar{z}_1\bar{z}_2^2 z_3) &= 2\tau_1\sigma_2 \\
\Re(z_2\bar{z}_2^3 z_3) &= 2\tau_2\sigma_1 \\
\Im(z_2\bar{z}_2^3 z_3) &= -2\tau_2\sigma_2 \\
\Re(z_2^2 z_3\bar{z}_3^2) &= 2\tau_3\sigma_1 \\
\Im(z_2^2 z_3\bar{z}_3^2) &= 2\tau_3\sigma_2.
\end{aligned}$$

These nine basic invariants generate the ring of functions invariant under the flow defined by H_0^0. Then the normal forms up to fifth order will consist of the invariants τ_1, τ_2, τ_3 and the additional real resonant terms $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$. For real constants $c_i, i = 1, ..., 18$ the resulting normal form takes the form

$$\begin{aligned}\mathcal{H}_5 = 2\tau_1 + 3\tau_2 + 6\tau_3 + c_1\sigma_1 + c_2\sigma_2 + c_3\tau_1^2 + c_4\tau_2^2 + c_5\tau_3^2 + c_6\tau_1\tau_2 + c_7\tau_2\tau_3 + c_8\tau_1\tau_3 + c_9\sigma_3 + c_{10}\sigma_4 + ...\\ c_{11}\sigma_5 + c_{12}\sigma_6 + c_{13}\tau_1\sigma_1 + c_{14}\tau_1\sigma_2 + c_{15}\tau_2\sigma_1 + c_{16}\tau_2\sigma_2 + c_{17}\tau_3\sigma_1 + c_{18}\tau_3\sigma_2.\end{aligned}$$

In the following subsections we will investigate the normal forms and apply it to a Hamiltonian that has cubic part $H_1^0(x, y) = \frac{x_2}{2}\left(x_1^2 + x_2x_3\right)$ and no higher order terms.

3.1 The cubic normal form

For constants c_1 and c_2 the cubic normal form takes the form

$$\mathcal{H}_3 = 2\tau_1 + 3\tau_2 + 6\tau_3 + c_1\sigma_1 + c_2\sigma_2.$$

with

$$\sigma_1 = \Re(z_2^2\bar{z}_3), \quad \sigma_2 = \Im(z_2^2\bar{z}_3).$$

If $(c_1, c_2) \neq (0, 0)$ we can perform a rotation and a scaling such that the normal form simply becomes

$$\mathcal{H}_3 = 2\tau_1 + 3\tau_2 + 6\tau_3 + \sigma_1.$$

As always, H_0^0 and $\mathcal{H}$ are integrals of the normal form system. Denote these by $F_1 = H_0^0$ and $F_2 = \mathcal{H}_3$. Now note that the third order normal form contains only one resonant cubic monomial (and its complex conjugate) which does not depend on z_1. Therefore we get the additional integral $F_3 = \tau_1$ since

$$\{\tau_1, \mathcal{H}_3\} = 0 \quad .$$

This follows from the Poisson bracket relations given in the introduction. All Poisson brackets are 0. Also $\{\tau_1, H_0^0\} = 0$ by the same argument hence all integrals are in involution. The integrals are independent since if $(c_1, c_2) \neq (0, 0)$, then

$$dF_1 \wedge dF_2 \wedge dF_3 \neq 0.$$

From this we can conclude it is an integrable system since we have three functionally independent integrals F_1, F_2, F_3. We cannot immediately conclude the system is non-degenerate since the cubic normal form now contains the terms σ_1 and σ_2 which cannot be written in our original action variables τ_1, τ_2 and τ_3. Therefore we do not know the exact frequency map and cannot easily check the non-degeneracy condition.

In the remainder of this subsection we will consider a Hamiltonian $H = H_0^0 + H_1^0$ that has cubic part $H_1^0(x, y) = \frac{x_2}{2}\left(x_1^2 + x_2x_3\right)$ and no higher order terms. For the resulting dynamics for the cubic normal form it does not really matter if we use this example or consider the general cubic normal form. But the example will later give useful results for the quartic normal form. Using that $x_i = \frac{1}{2}(z_i + \bar{z}_i)$ we can write the cubic Hamiltonian in the following way:

$$\begin{aligned} H_1^0 &= \frac{x_2}{2}\left(x_1^2 + x_2x_3\right)\\ &= \frac{1}{2}(x_1^2x_2 + x_2^2x_3)\\ &= \frac{1}{2}\left(\frac{(z_1+\bar{z}_1)^2}{4}\cdot\frac{z_2+\bar{z}_2}{2} + \frac{(z_2+\bar{z}_2)^2}{4}\cdot\frac{z_3+\bar{z}_3}{2}\right)\\ &= \frac{1}{16}\left((z_1^2 + 2z_1\bar{z}_1 + \bar{z}_1^2)(z_2+\bar{z}_2) + (z_2^2 + 2z_2\bar{z}_2 + \bar{z}_2^2)(z_3+\bar{z}_3)\right)\\ &= \frac{1}{16}\left(z_1^2z_2 + 2z_1\bar{z}_1z_2 + \bar{z}_1^2z_2 + z_1^2\bar{z}_2 + 2z_1\bar{z}_1\bar{z}_2 + \bar{z}_1^2\bar{z}_2 \dots\right.\\ &\qquad \left. + z_2^2z_3 + 2z_2\bar{z}_2z_3 + \bar{z}_2^2z_3 + z_2^2\bar{z}_3 + 2z_2\bar{z}_2\bar{z}_3 + \bar{z}_2^2\bar{z}_3\right). \end{aligned} \tag{1}$$

This term contains the resonant term $\frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3)$. All other terms are non-resonant and can be eliminated. We will do this explicitly by defining some Hamiltonian W which consists of third order monomials and perform a coordinate transformation by the time-1 flow along the vector field generated by W. We want to choose W in such a way that $H_1^0 - \frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3) = -\{H_0^0, W\}$. Then the transformed Hamiltonian becomes

$$\begin{aligned}\mathcal{H} = H \circ \varphi_{t=1}^W &= H + \{H, W\} + \frac{1}{2}\{\{H, W\}, W\} + \text{h.o.t.} \\ &= H_0^0 + H_1^0 + \{H_0^0 + H_1^0, W\} + \frac{1}{2}\left\{\{H_0^0 + H_1^0, W\}, W\right\} + \text{h.o.t.} \\ &= H_0^0 + \frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3) + \frac{1}{2}\{H_1^0, W\} + \frac{1}{2}\left\{\frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3), W\right\} + \frac{1}{2}\left\{\{H_1^0, W\}, W\right\} + \text{h.o.t.}.\end{aligned} \tag{2}$$

We can immediately conclude that the third order normal form will be

$$\mathcal{H}_3 = H_0^0 + \frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3) \quad .$$

This follows from the fact that the Poisson bracket lowers order by 2 and hence $\{H_1^0, W\}$ and $\left\{\frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3), W\right\}$ are of fourth order, $\{\{H_1^0, W\}, W\}$ is of fifth order, etc.
To find W, consider all non-resonant monomials in H_1^0 and compute the Poisson bracket with H_0^0. We show the following bracket explicitly:

$$\begin{aligned}\{z_1^2 z_2, H_0^0\} &= 2i(-4-3) z_1^2 z_2 \\ &= -14 i z_1^2 z_2.\end{aligned}$$

From this we can conclude that $\left\{\frac{1}{16} \cdot \frac{1}{14} i z_1^2 z_2, H_0^0\right\} = \frac{1}{16} z_1^2 z_2$. Performing similar calculations for the remaining non-resonant monomials gives us the function W:

$$\begin{aligned}W = \frac{i}{16}\Bigg(&\frac{z_1^2 z_2}{14} + \frac{z_1 \bar{z}_1 z_2}{3} - \frac{\bar{z}_1^2 z_2}{2} + \frac{z_1^2 \bar{z}_2}{2} - \frac{z_1 \bar{z}_1 \bar{z}_2}{3} - \frac{\bar{z}_1^2 \bar{z}_2}{14} \dots \\ &+ \frac{z_2^2 z_3}{24} + \frac{z_2 \bar{z}_2 z_3}{6} - \frac{z_2 \bar{z}_2 \bar{z}_3}{6} - \frac{\bar{z}_2^2 \bar{z}_3}{24}\Bigg).\end{aligned} \tag{3}$$

After this coordinate transformation we are left with the third order normal form $\mathcal{H}_3 = H_0^0 + \frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3)$.

Now that we have obtained this normal form we can study the dynamics of the corresponding Hamiltonian system. Recall that $\sigma_1 = \Re(z_2^2 \bar{z}_3) = \frac{1}{2}(z_2^2 \bar{z}_3 + \bar{z}_2^2 z_3)$. Then the third order normal form becomes

$$\mathcal{H}_3 = H_0^0 + \frac{1}{8}\sigma_1 \quad .$$

where we have the five basic invariants $(\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2)$. We want to reduce dimensions to determine the dynamics. First notice that τ_1 is a constant of motion of $\mathcal{H}_3$ so set $\tau_1 = c$ for now. This corresponds to reducing the phase space by the S^1 action induced by the conserved quantity τ_1. To study the dynamics we introduce a detuning parameter λ into the Hamiltonian as follows

$$\mathcal{H}_3^\lambda = 2c + (3+\lambda)\tau_2 + 6\tau_3 + \frac{1}{8}\sigma_1 \quad .$$

We could get rid of the constant c since this would not influence the dynamics but since it is also τ_1 we keep it in the equation.

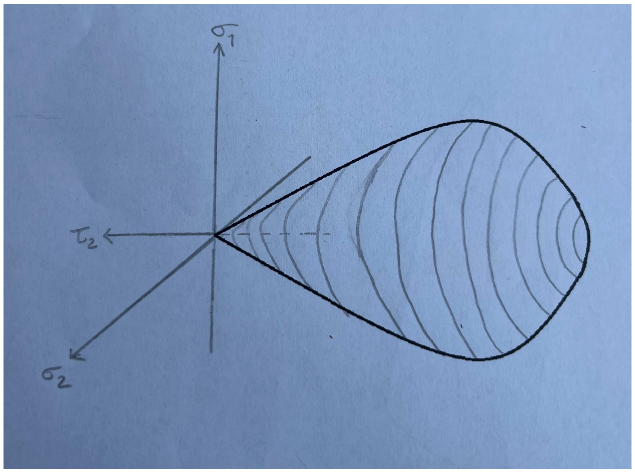


Figure 1: Surface of revolution corresponding to the reduced phase space. The lines on the surface are geodesics to show the curvature.

Then between the four invariants that are left we have the syzygy

$$\begin{aligned}\sigma_1^2 + \sigma_2^2 &= \frac{1}{4}(z_2^2\bar{z}_3 + \bar{z}_2^2 z_3)^2 - \frac{1}{4}(z_2^2\bar{z}_3 - \bar{z}_2^2 z_3)^2 \\ &= \frac{1}{4}(4z_2^2\bar{z}_3\bar{z}_2^2 z_3) \\ &= (z_2\bar{z}_2)^2 z_3\bar{z}_3 \\ &= 8\tau_2^2\tau_3 \quad .\end{aligned}$$

This reduces the dimension again by one. We also have the conserved quantity $\eta = H_0^0 = 2c + 3\tau_2 + 6\tau_3$ and hence we can write

$$\tau_3 = \frac{\eta - 2c - 3\tau_2}{6}, \tag{4}$$

further reducing the dimension by one. Plugging this into the syzygy yields

$$\sigma_1^2 + \sigma_2^2 = \frac{4\tau_2^2(\eta - 2c - 3\tau_2)}{3}.$$

Furthermore we have the relations

$$\tau_2 \geq 0, \quad \tau_3 \geq 0 \quad ,$$

from which we can deduce that $0 \leq \tau_2 \leq \frac{\eta - 2c}{3}$. What we are left with is the following reduced phase space:

$$\mathcal{P}_{c,\eta} = \left\{ (\tau_2, \sigma_1, \sigma_2) \in \mathbb{R}^3 \;\middle|\; 0 \leq \tau_2 \leq \frac{\eta - 2c}{3},\ \sigma_1^2 + \sigma_2^2 = \frac{4\tau_2^2(\eta - 2c - 3\tau_2)}{3} \right\} \quad .$$

This is a two dimensional surface which consists of circles in the (σ_1, σ_2) plane where the radius of the circles depends on τ_2. If we set $\tau_1 = c = 0$, then the origin in the reduced phase space becomes the normal 3-mode. This point is an equilibrium. In the original system this would correspond to a periodic orbit. The point on the surface at $(\tau_2, \sigma_1, \sigma_2) = (\frac{\eta - 2c}{3}, 0, 0)$ corresponds to the normal 2-mode. As we will see in the following parts, this is not automatically an equilibrium. From equation (4) we conclude that we can only consider the normal 1-mode if $\tau_1 = c = \frac{\eta}{2}$. Then the whole reduced phase space would reduce to a single point at the origin and is also an equilibrium which corresponds to a periodic orbit in the original system. This surface is smooth everywhere except at the origin. Because of isotropy, the origin is always an equilibrium. Now we substitute equation (4) into $\mathcal{H}_3^\lambda$ to find that

$$\begin{aligned}\mathcal{H}_3^\lambda &= 2c + (3+\lambda)\tau_2 + \eta - 2c - 3\tau_2 + \frac{1}{8}\sigma_1 \\ &= \eta + \lambda\tau_2 + \frac{1}{8}\sigma_1 \quad .\end{aligned}$$

The level sets of this Hamiltonian are given by two dimensional planes in $\mathbb{R}^3$ where the orientation of these planes depends on the parameter λ. To draw the phase portraits we need to find the intersections

$$\mathcal{P}_{c,\eta} \cap \mathcal{H}_3^\lambda \quad .$$

To figure out in which direction the orbits go we need to compute a single Poisson bracket:

$$\begin{aligned}\dot{\tau}_2 &= \{\tau_2, \mathcal{H}_3^\lambda\} \\ &= \frac{1}{8}\{\tau_2, \sigma_1\} \\ &= \frac{1}{32}\left(\{z_2\bar{z}_2, z_2^2\bar{z}_3\} + \{z_2\bar{z}_2, \bar{z}_2^2 z_3\}\right) \\ &= \frac{1}{32}\left(z_2\{\bar{z}_2, z_2^2\bar{z}_3\} + \bar{z}_2\{z_2, \bar{z}_2^2 z_3\}\right) \\ &= \frac{1}{32}\left(z_2\bar{z}_3\{\bar{z}_2, z_2^2\} + \bar{z}_2 z_3\{z_2, \bar{z}_2^2\}\right) \\ &= \frac{1}{32}\left(2z_2^2\bar{z}_3\{\bar{z}_2, z_2\} + 2\bar{z}_2^2 z_3\{z_2, \bar{z}_2\}\right) \\ &= \frac{1}{8}(iz_2^2\bar{z}_3 - i\bar{z}_2^2 z_3) \\ &= -\frac{1}{4}\left(\frac{z_2^2\bar{z} - \bar{z}_2^2 z_3}{2i}\right) = -\frac{1}{4}\sigma_2 \quad .\end{aligned} \tag{5}$$

In the third and fourth line we used Leibniz' rule together with the fact that the only non-zero bracket between complex variables is $\{z_i, \bar{z}_i\}$ for $i = 1, 2, 3$. We conclude that τ_2 moves in the negative direction if $\sigma_2 > 0$ and τ_2 moves in the positive direction if $\sigma_2 < 0$. The planes of the Hamiltonian have slope -8λ in the (τ_2, σ_1) plane so the planes of the Hamiltonian are tangent to $\mathcal{P}_{c,\eta}$ at $(0,0,0)$ when $\frac{d}{d\tau_2}\sigma_1|_{\tau_2=0} = -8\lambda$. In the (τ_2, σ_1) plane the surface is given by

$$\sigma_1 = \pm\sqrt{\frac{4\tau_2^2(\eta - 2c - 3\tau_2)}{3}},$$

so the planes are tangent to $\mathcal{P}_{c,\eta}$ at $(0,0,0)$ precisely when

$$\begin{aligned}\lambda &= -\frac{1}{8}\frac{d}{d\tau_2}\left[\pm\sqrt{\frac{4\tau_2^2(\eta - 2c - 3\tau_2)}{3}}\right]_{\tau_2=0} \\ &= \pm\frac{1}{12}\sqrt{3(\eta - 2c)}.\end{aligned}$$

At these values of λ the equilibrium at $(0,0,0)$ undergoes a Hamiltonian flip/period doubling bifurcation. To draw the phase portrait, we will draw the orbit lines by projecting the surface when $\sigma_2 < 0$ to the plane where $\sigma_2 = 0$. This projection is visualized in figure (2). Interestingly, if $\lambda \to \pm\infty$, then the normal 2-mode does become an equilibrium in the reduced phase space and a periodic orbit in the original system. The phase portraits for interesting values of λ can be found in figure (3). We have given only phase portraits for non-positive λ but the phase portraits extend easily to positive λ by reversing the direction of rotation of the level sets of the Hamiltonian.

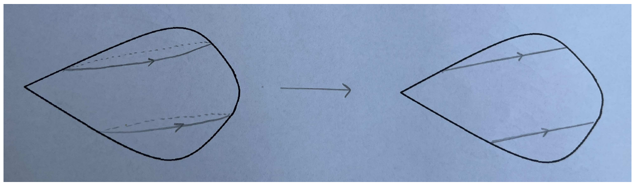
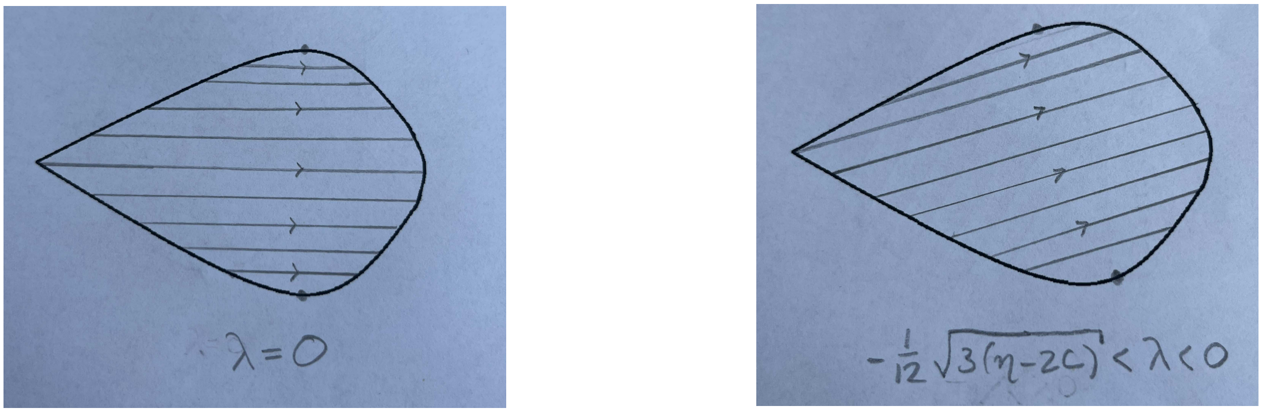
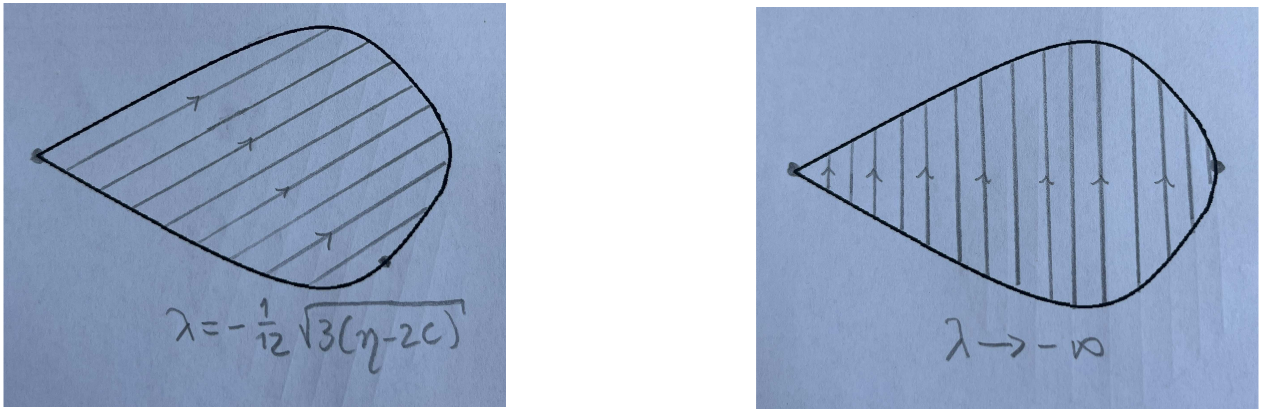


Figure 2: Projection of the orbits on the surface $\mathcal{P}_{c,\eta}$ for $\sigma_2 < 0$ to the plane $(\tau_2, \sigma_1, 0)$.



(a) $\lambda = 0$

(b) $-\frac{1}{12}\sqrt{3(\eta - 2c)} < \lambda < 0$



(c) $\lambda = -\frac{1}{12}\sqrt{3(\eta - 2c)}$

(d) $\lambda \to -\infty$

Figure 3: Phase portraits for specific values of λ.

3.2 The quartic normal form

From the introduction of this section we know that the fourth order resonant terms are the monomials

$$\tau_1^2, \quad \tau_2^2, \quad \tau_3^2, \quad \tau_1\tau_2, \quad \tau_1\tau_3, \quad \tau_2\tau_3 \quad ,$$

and

$$z_1^3\bar{z}_3, \quad \bar{z}_1^3 z_3 \quad .$$

Define $\sigma_3 = \Re(z_1^3\bar{z}_3)$ and $\sigma_4 = \Im(z_1^3\bar{z}_3)$. Then the fourth order normal form will take the following form

$$\mathcal{H}_4 = 2\tau_1 + 3\tau_2 + 6\tau_3 + c_1\sigma_1 + c_2\sigma_2 + c_3\tau_1^2 + c_4\tau_2^2 + c_5\tau_3^2 + c_6\tau_1\tau_2 + c_7\tau_1\tau_3 + c_8\tau_2\tau_3 + c_9\sigma_3 + c_{10}\sigma_4.$$

If $(c_1, c_2) \neq 0$ and $(c_9, c_{10}) \neq 0$ we can perform a rotation and a scaling such that the normal form becomes

$$\mathcal{H}_4 = 2\tau_1 + 3\tau_2 + 6\tau_3 + \sigma_1 + c_3\tau_1^2 + c_4\tau_2^2 + c_5\tau_3^2 + c_6\tau_1\tau_2 + c_7\tau_1\tau_3 + c_8\tau_2\tau_3 + c_9\sigma_3.$$

Note that we also obtain the following syzygy

$$\sigma_3^2 + \sigma_4^2 = 16\tau_1^3\tau_3.$$

The quartic normal form is in general not immediately integrable since τ_1 is not necessarily conserved anymore. We see this by the following:

$$\{\tau_1, \sigma_3\} = -3\sigma_4 \neq 0,$$

by a similar computation as in equation (5) for $\dot{\tau}_2$. So there is no obvious third integral. The system can be integrable if the resonant terms σ_3, σ_4 do not appear in the quartic normal form. Then τ_1 is still an integral of the system. Of course it is still possible that the general quartic normal form is integrable with a non-trivial third integral. We again cannot check non-degeneracy explicitly. However with every higher order normal form it is expected that non-degeneracy holds with a higher probability than before.

To study the dynamics of the quartic normal form we will follow a similar approach as in [HMSV21]. The goal will be to conclude if the normal modes are equilibria. We compute all Poisson brackets between the basic invariants. These can be found in table 1 and 2. Note that in table 2 some Poisson brackets equate to the invariants σ_5 and σ_6. This signifies that the basic invariants up to fifth order generate the ring of invariant functions under the flow of H_0^0.

$\{\downarrow, \rightarrow\}$	σ_1	σ_2	σ_3	σ_4
τ_1	0	0	$-3\sigma_4$	$3\sigma_3$
τ_2	$-2\sigma_2$	$2\sigma_1$	0	0
τ_3	σ_2	$-\sigma_1$	σ_4	$-\sigma_3$

Table 1: The Poisson brackets between the basic invariants $\tau_i,\ i = 1, 2, 3$ and $\sigma_j,\ j = 1, 2, 3, 4$.

$\{\downarrow, \rightarrow\}$	σ_1	σ_2	σ_3	σ_4
σ_1	0	$4\tau_2(4\tau_3 - \tau_2)$	σ_6	$-\sigma_5$
σ_2	$-4\tau_2(4\tau_3 - \tau_2)$	0	σ_5	σ_6
σ_3	$-\sigma_6$	$-\sigma_5$	0	$8\tau_1^2(9\tau_3 - \tau_1)$
σ_4	σ_5	$-\sigma_6$	$-8\tau_1^2(9\tau_3 - \tau_1)$	0

Table 2: The Poisson brackets between the basic invariants $\sigma_i,\ i = 1, 2, 3, 4$.

From the syzygys and the Poisson brackets we can conclude that $\{\tau_1 = 0\}$ and $\{\tau_2 = 0\}$ are invariant sets. Take for instance $\{\tau_1 = 0\}$. By the syzygys we know that $\sigma_3 = \sigma_4 = 0$. Furthermore by the chain rule, Leibniz rule and the Poisson brackets we have that

$$\begin{aligned}
\dot{\tau}_1 &= \{\tau_1, \mathcal{H}_4\} \\
&= \sum_{i=1}^{3} \{\tau_1, \tau_j\} \frac{\partial \mathcal{H}_4}{\partial \tau_i} + \sum_{j=1}^{4} \{\tau_1, \sigma_j\} \frac{\partial \mathcal{H}_4}{\partial \sigma_j} \\
&= \sum_{j=1}^{4} \{\tau_1, \sigma_j\} \frac{\partial \mathcal{H}_4}{\partial \sigma_j} \\
&= -3\sigma_4 \frac{\partial \mathcal{H}_4}{\partial \sigma_3} + 3\sigma_3 \frac{\partial \mathcal{H}_4}{\partial \sigma_4} = 0.
\end{aligned}$$

Similarly we can conclude that $\dot{\sigma}_3 = \dot{\sigma}_4 = 0$ where we use that $\tau_1 = 0$ if and only if $z_1 = 0$. A similar calculation holds for $\{\tau_2 = 0\}$ for which holds that $\sigma_1 = \sigma_2 = 0$ by the syzygy and $\dot{\tau}_2 = \dot{\sigma}_1 = \dot{\sigma}_2 = 0$. From this we can immediately conclude that the normal 3-mode $(\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0, 0, \tau_3, 0, 0, 0, 0)$ is an equilibrium since all derivatives of the basic invariants (including $\dot{\tau}_3$) are 0.
The set $\{\tau_3 = 0\}$ is not an invariant set. For this set we would have that $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, However

$\dot{\sigma}_i \neq 0,\ i = 1,2,3,4$. This follows since the brackets $\{\sigma_1, \sigma_2\}$ and $\{\sigma_3, \sigma_4\}$ are not necessarily 0 if $\tau_3 = 0$. It can also be seen that the normal 3-mode is an equilibrium in the following way; on the invariant set $\{\tau_1 = 0\}$ we can view the reduced phase space in a similar manner as in the case for the cubic normal form. On this set $\sigma_3 = \sigma_4 = 0$ so we are left with the basic invariants $(\tau_2, \tau_3, \sigma_1, \sigma_2)$. Using the syzygy $\sigma_1^2 + \sigma_2^2 = 8\tau_2^2\tau_3$ and the conserved quantity $\eta = 3\tau_2 + 6\tau_3$ we can construct a reduced phase space in the variables $(\tau_2, \sigma_1, \sigma_2)$ and intersect this with the level sets of the Hamiltonian. On the reduced phase space $(\tau_2, \sigma_1, \sigma_2) = (0,0,0)$ is always an equilibrium. We can thus conclude that the normal 3-mode is an equilibrium. Similar reasoning can be used on the invariant set $\{\tau_2 = 0\}$ to conclude the normal 3-mode is an equilibrium. The reduced phase space looks slightly different from the 1:2 resonance, it does not grow as fast around the origin. A sketch of what this would look like can be found in figure (4). Here τ_3 corresponds to the singular point with isotropy $\mathbb{Z}_6$. The singular part corresponding to the invariant set $\{\tau_1 = 0\}$ has isotropy $\mathbb{Z}_3$ and the singular part corresponding to the invariant set $\{\tau_2 = 0\}$ has isotropy $\mathbb{Z}_2$. On the sets $\{\tau_i > 0\}$ it is not possible to reduce as much. Then the σ_i are not immediately 0 from the syzygys. On this set it is much more difficult to obtain results about the dynamics.

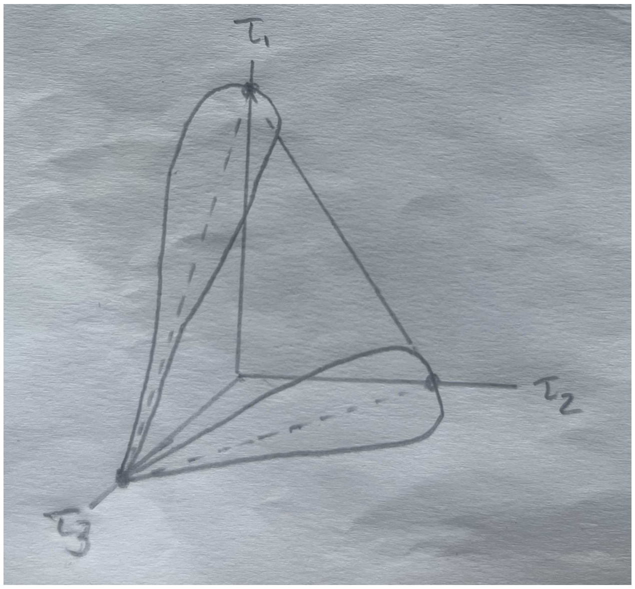


Figure 4: The simplex corresponding to the quartic normal form drawn in the action space. On the invariant sets $\{\tau_1 = 0\}$ and $\{\tau_2 = 0\}$ the singular parts can be found corresponding to the reduced two dimensional phase space.

To illustrate this, consider the same Hamiltonian $H = H_0^0 + H_1^0$ we considered in the cubic case that has cubic part $H_1^0(x,y) = \frac{x_2}{2}\left(x_1^2 + x_2 x_3\right)$. To find the quartic normal form, recall equations (1), (2) and (3). The fourth order terms in the cubic normal form are given by

$$\frac{1}{2}\{H_1^0, W\} + \frac{1}{2}\left\{\frac{1}{16}(\bar{z}_2^2 z_3 + z_2^2 \bar{z}_3), W\right\}.$$

To find which monomials are resonant in this term, we could compute all 120 Poisson brackets between the 12 monomials in H_1^0 and the 10 monomials in W. To avoid this computation we will consider a different strategy; Consider general monomials $f, g \in \mathcal{G}_3^{\mathbb{C}}$. What are the possible forms these monomials can take such that their bracket produces a resonant term? To investigate this, recall that by the coordinate transformation to complex coordinates the canonical Poisson bracket takes the following form:

$$\{f, g\} = -2i \sum_{j=1}^{3} \left(\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j}\frac{\partial g}{\partial z_j} \right).$$

Resonant monomial	Possible product for $j = 1, 2, 3$	Could this occur?
τ_1^2	$z_1^2 \bar{z}_1^2 z_j \bar{z}_j$	yes if $j = 2$, two options
τ_2^2	$z_2^2 \bar{z}_2^2 z_j \bar{z}_j$	yes if $j = 3$, two options
τ_3^2	$z_3^2 \bar{z}_3^2 z_j \bar{z}_j$	no
$\tau_1\tau_2$	$z_1 \bar{z}_1 z_2 \bar{z}_2 z_j \bar{z}_j$	yes if $j = 1$, two options
$\tau_1\tau_3$	$z_1 \bar{z}_1 z_3 \bar{z}_3 z_j \bar{z}_j$	no
$\tau_2\tau_3$	$z_2 \bar{z}_2 z_3 \bar{z}_3 z_j \bar{z}_j$	yes if $j = 2$, two options
$z_1^3 \bar{z}_3$	$z_1^3 \bar{z}_3 z_j \bar{z}_j$	no
$\bar{z}_1^3 z_3$	$\bar{z}_1^3 z_3 z_j \bar{z}_j$	no

Table 3: In the left column are all possible quartic resonant monomials displayed. If a possible product in the middle column cannot occur, then certainly the corresponding resonant term is not in the normal form. If it can occur, further computation is needed to show it is in the normal form.

Suppose we want the bracket to produce at least one monomial $m \in \mathcal{G}_4^{\mathbb{C}}$. Consider the product fg. If the bracket produces a monomial m, then we know that for some $j \in \{1, 2, 3\}$ the product fg must be of the form $mz_j\bar{z}_j$ since the bracket differentiates once with respect to z_j and once with respect to $\bar{z}_j$. The reverse implication is not necessarily true. We can however use the negation of this statement to conclude that the bracket cannot produce a monomial m if the product fg is not of the form $mz_j\bar{z}_j$. We can check for every resonant monomial which fg corresponds to it in the following table: From table (3) we can conclude that the only resonant monomials that can possibly appear in the fourth order terms are $\tau_1^2, \tau_1\tau_2, \tau_2^2$ and $\tau_2\tau_3$. What remains is computing the remaining Poisson brackets that might produce those resonant terms to check if they actually do. Note in table (3) that there are actually only two possible products, namely $z_1^2\bar{z}_1^2 z_2\bar{z}_2$ and $z_2^2\bar{z}_2^2 z_3\bar{z}_3$. The Poisson brackets are as follows:

- $z_1^2\bar{z}_1^2 z_2\bar{z}_2$: This is the product of the monomials $\bar{z}_1^2 z_2 \cdot z_1^2\bar{z}_2$ or $z_1\bar{z}_1 z_2 \cdot z_1\bar{z}_1\bar{z}_2$. These yield the Poisson brackets $\{\bar{z}_1^2 z_2, z_1^2\bar{z}_2\} = 8iz_1\bar{z}_1 z_2\bar{z}_2 - 2iz_1^2\bar{z}_1^2$ and $\{z_1\bar{z}_1 z_2, z_1\bar{z}_1\bar{z}_2\} = -2iz_1^2\bar{z}_1^2$.
- $z_2^2\bar{z}_2^2 z_3\bar{z}_3$: This is the product of the monomials $z_2^2 z_3 \cdot \bar{z}_2^2\bar{z}_3$ or $z_2\bar{z}_2 z_3 \cdot z_2\bar{z}_2\bar{z}_3$. These yield the Poisson brackets $\{z_2^2 z_3, \bar{z}_2^2\bar{z}_3\} = -8iz_2\bar{z}_2 z_3\bar{z}_3 - 2iz_2^2\bar{z}_2^2$ and $\{z_2\bar{z}_2 z_3, z_2\bar{z}_2\bar{z}_3\} = -2iz_2^2\bar{z}_2^2$.

Hence we can conclude that the resonant monomials $\tau_1^2, \tau_2^2, \tau_1\tau_2$ and $\tau_2\tau_3$ are present in the quartic normal form. Using this approach we have reduced the number of Poisson bracket computations from 120 to just 4.

In particular note that the terms σ_3 and σ_4 do not appear. This means that for this specific normal form $\tau_1 = c$ is still a conserved quantity and this quartic normal form defines an integrable system. Using equation (4) the quartic normal form for constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with detuning parameter λ takes the following form:

$$\begin{aligned}
\mathcal{H}_4^\lambda &= 2\tau_1 + (3+\lambda)\tau_2 + 6\tau_3 + \frac{1}{8}\sigma_1 + c_1\tau_1^2 + c_2\tau_1\tau_2 + c_3\tau_2^2 + c_4\tau_2\tau_3 \\
&= 2c + (3+\lambda)\tau_2 + 6\left(\frac{\eta - 2c - 3\tau_2}{6}\right) + \frac{1}{8}\sigma_1 + c_1c^2 + c_2c\tau_2 + c_4\tau_2\frac{\eta - 2c - 3\tau_2}{6} + c_3\tau_2^2 \\
&= \eta + c_1c^2 + \frac{1}{8}\sigma_1 + \left(\lambda + c_2c + \frac{c_4(\eta - 2c)}{6}\right)\tau_2 + \left(c_3 - \frac{c_4}{2}\right)\tau_2^2.
\end{aligned}$$

We thus see that the level sets of the Hamiltonian are parabolas in the (τ_2, σ_1) plane. Since τ_1 is still a conserved quantity, we can perform the same reduction in the phase space as we did for the cubic normal form. The only difference will be the orbits since the Hamiltonian has changed from being linear to quadratic. An example of a possible phase portrait can be found in figure (5). The bifurcations that occur when λ is tuned become much more complex than in the cubic case since the intersecting level sets are now parabolas of which the shape depends greatly on many parameters and constants

3.3 Higher order normal forms

Of course we can continue the process of normalizing to obtain higher order normal forms. When normalizing up to fifth order we find the resonant terms including the basic invariants σ_5, σ_6. For higher orders,

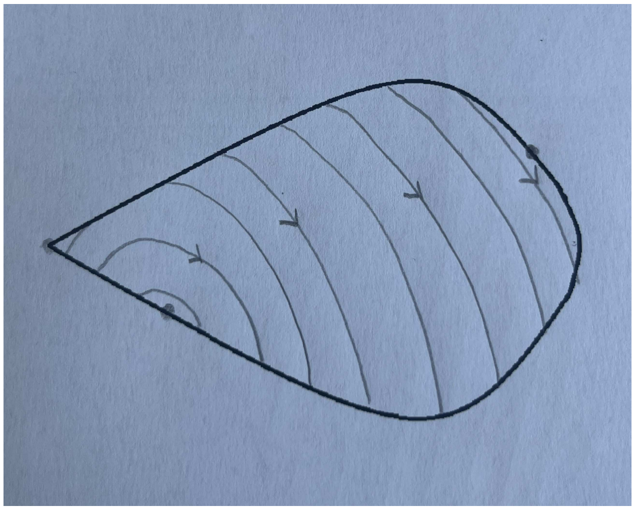


Figure 5: A possible phase portrait for the system defined by the quartic normal form.

all resonant monomials will be combinations of the 9 basic invariants we have found. The resulting space is however not 9 dimensional since these basic invariants are related through many syzygys. For the fifth order normal form, also many ideas from [HMSV21] can be applied. All Poisson brackets could be computed between all basic invariants to show that the rank of the matrix $(\{\tau_i, \sigma_j\})_{i,j} \leq 4$. If for instance the rank of this matrix equals zero, this could give information about equilibria and normal modes.

The resulting Hamiltonian will describe the original system more closely but it becomes a more complicated function. The probability of the resulting system being integrable is then much smaller. If our cubic normal form turns out to be non-degenerate then we can apply the KAM-theorem to it since it is definitely integrable. Then there is no real need to normalize further since the dynamics are mostly understood.

4 Conclusions

We studied the 2:3:6 resonance in normal forms of Hamiltonian systems. In order to do this we started by reviewing some theory on Hamiltonian systems and normal forms. We then applied this to obtain the cubic normal form. This normal form defined in fact an integrable Hamiltonian system and hence we could study the dynamics intensively. The reduced phase space became a compact surface embedded in $\mathbb{R}^3$ on which we could draw the orbits by intersecting this surface with the Hamiltonian level sets. For the quartic normal form we could not immediately conclude integrability. We could give a general statement about the normal 3-mode being an equilibrium. For the specific example of the Hamiltonian with cubic part $H_1^0 = \frac{x_2}{2}\left(x_1^2 + x_2 x_3\right)$ the quartic normal form actually defines an integrable Hamiltonian system and we can discuss the dynamics in a similar way as for the cubic normal form. When normalizing further it becomes significantly harder to say anything meaningful about the dynamics of the system defined by the normal form.

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