# INLDS Essay: Uniqueness of the limit cycle near Hopf-bifurcation

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## Summery

In this essay, we analyze the supercritical Hopf-bifurcation. We will prove that close to the origin, this system has one unique limit cycle. In section two we will prove this by using the Poincaré-map, which we derive in section one. In section three, we will give another proof, using the Bendixson-Dulac theorem and the Poincaré-Bendixson theorem.


## Introduction

The smooth orbital normal form for the supercritical Hopf bifurcation is defined as

$$\dot{w} = (\alpha + i)w - w|w|^2 + O(|w|^4), \quad w \in \mathbb{C}, \tag{1}$$

where the $O(|w|^4)$-terms can smoothly depend on $\alpha \in \mathbb{R}$. [1]

For our essay, it is more useful to write this system in polar coordinates. Herefore, we write $w = \rho e^{i\phi}$, with $\dot{w} = \dot{\rho} e^{i\phi} + i\rho\dot{\phi} e^{i\phi}$. If we substitute this in (1), we get the following equation:

$$\dot{\rho} e^{i\phi} + i\rho\dot{\phi} e^{i\phi} = \alpha\rho e^{i\phi} - \rho^3 e^{i\phi} + i\rho e^{i\phi} + O(|w|^4). \tag{2}$$

This gives rise to the equations for the real and imaginary parts: $\dot{\rho} e^{i\phi} = \alpha\rho e^{i\phi} - \rho^3 e^{i\phi} + O(|w|^4)$ and $\rho\dot{\phi} e^{i\phi} = \rho e^{i\phi} + O(|w|^4)$. In this essay, we will focus on the points on the complex plane with small $\rho > 0$, unless stated otherwise. Therefore we can now use (2) to write our system in polar coordinates:

$$\begin{cases} \dot{\rho} &= \rho(\alpha - \rho^2) + \Phi(\rho, \phi) \\ \dot{\phi} &= 1 + \Psi(\rho, \phi) \end{cases} \tag{3}$$

We note that $\Phi(\rho, \phi) = O(|\rho|^4)$ and $\Psi = O(|\rho|^3)$. The $\alpha$-dependence of these smooth functions is not indicated to simplify notations. At this point we emphasize that $|w| = \rho$ and we will use the notations $O(|w|^k), O(|\rho|^k),\ k \in \mathbb{N}$ interchangeably.

## 1 Cubic Taylor expansion of the Poincaré map and it's dependence on the higher order terms of the supercritical Hopf bifurcation.

In this section, we aim to find a Taylor expansion of the parameter-dependent Poincaré map of system (1) and to analyze it's dependence on the $O(|w|^4)$ terms.

We will look at an orbit of (3), which starts at $(\rho, \phi) = (\rho_0, 0)$, with very small $\rho_0$. For this orbit we will use the representation $\rho = \rho(\phi; \rho_0),\ \rho_0 = \rho(0; \rho_0)$. A visual representation is included in figure 1 below.

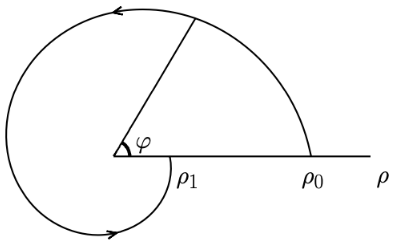


Figure 1: Visual representation of the Poincaré map for system (3), picture is from the book "Elements of Applied Bifurcation Theory" [2].

From (3), we know that the orbit $\rho(\rho_0, 0)$ satisfies

$$\frac{d\rho}{d\phi} = \frac{\rho(\alpha - \rho^2) + \Phi(\rho, \phi)}{1 + \Psi(\rho, \phi)}. \tag{4}$$

For convenience, we write the smooth function $\frac{1}{1+\Psi(\rho,\phi)}$ as it's Taylor expansion around $\psi = 0 : \frac{1}{1+\Psi} = 1 - \Psi + \Xi(\Psi)$, where $\Xi(\Psi)$ is a smooth function of order $O(|\Psi|^2)$. Therefore we write

$$\frac{d\rho}{d\phi} = (\rho(\alpha - \rho^2) + \Phi)(1 - \Psi + \Xi(\Psi)) = \rho(\alpha - \rho^2) + R(\rho, \phi), \tag{5}$$

where $R(\rho, \phi)$ is a smooth function of order $O(|\rho|^3)$. We recall that $\Phi(\rho, \phi) = O(|\rho|^4)$ and $\Psi = O(|\rho|^3)$. Therefore $\Xi(\Psi) = O(|\Psi|^2) = O(|\rho|^6)$ and we conclude that $R(\rho, \phi) = O(|\rho|^4)$.

In the next step we will write the cubic Taylor expansion for $\rho(\phi, \rho_0)$ at $\rho_0 = \rho(\phi, 0) \equiv 0$ :

$$\rho = u_1(\phi)\rho_0 + u_2(\phi)\rho_0^2 + u_3(\phi)\rho_0^3 + O(|\rho_0|^4). \tag{6}$$

The next step is to find the expressions for $u_i(\phi),\ i = 1, 2, 3$, and look at their dependency on the $O(|w|^4)$ terms of (1). To do so, we substitute (6) into (5):

$$\begin{aligned} u_1'(\phi)\rho_0 + u_2'(\phi)\rho_0^2 + u_3'(\phi)\rho_0^3 + O(|\rho_0|^4) &= \alpha(u_1\rho_0 + u_2\rho_0^2 + u_3\rho_0^3 + O(|\rho_0|^4)) - u_1^3\rho_0^3 - O(|\rho|^4) \\ &= \alpha u_1\rho_0 + \alpha u_2\rho_0^2 + (\alpha u_3 - u_1^3)\rho_0^3 + O(|\rho_0|^4). \end{aligned}$$

This gives us three sets of linear differential equations. Before we write them down, we note that we have the initial conditions $u_1(0) = 1, u_2(0) = u_3(0) = 0$, since $\rho(0, \rho_0) = \rho_0$ : The first equation is $u_1' = \alpha u_1$, with solution $u_1(\phi) = e^{\alpha\phi}$. The second equation is $u_2' = \alpha u_2$, from which we find $u_2 \equiv 0$. Finally, $u_3' = \alpha u_3 - u_1^3$ so we conclude $u_3(\phi) = -2\pi$ if $\alpha = 0$ and $u_3(\phi) = e^{\alpha\phi}\frac{1-e^{2\alpha\phi}}{2\alpha}$ for $|\alpha| > 0$. At this point we make a small note that $\lim_{\alpha\to 0} e^{\alpha\phi}\frac{1-e^{2\alpha\phi}}{2\alpha} = -2\pi$ so the definition of $u_3(\phi)$ is consistant.

We look at the Poincaré mapping of (1) defined on the half-axis $\text{Re}(w) \geq 0$, near $w = 0$. Because $\dot{\phi} \approx 1$, the Poincaré map is defined by $\rho_0 \mapsto \rho_1 := \rho(2\pi, \rho_0)$. We can use (6) and the expressions for $u_1,\ i = 1, 2, 3$ to express the Poincaré map:

$$\rho_1(\rho_0) = e^{2\pi\alpha}\rho_0 + e^{2\pi\alpha}\left(\frac{1 - e^{4\alpha\pi}}{2\alpha}\right)\rho_0^3 + O(|\rho_0|^4), \quad \alpha \neq 0 \tag{7}$$

$$\rho_1(\rho_0) = e^{2\pi\alpha}\rho_0 - 2\pi\rho_0^3 + O(|\rho_0|^4), \quad \alpha = 0 \tag{8}$$

Because the expressions for $u_i,\ i = 1, 2, 3$, are independent of $R(\phi, \psi)$, we conclude that the Poincaré map above is independent of all $O(|\rho|^4)$ terms of the system (3) in Polar coördinates. Therefore the Poincaré map is also independent of the higher order $O(|w|^4)$ terms of (1).

## 2 Fixed point of the Poincaré map and limit cycles of the Hopf bifurcation

In this section we will prove that the Poincaré map has a unique stable positive fixed point when $\alpha > 0$. We will then use this to conclude that a unique stable limit cycle bifurcates from the origin in (1) independent of the $O(|w|^4)$–terms.

We will analyze the Poincaré-map for small $\rho_0, |\alpha|$. We start of by finding an expression for the fixed points $r$, such that $\rho_1(r) = r$. For $\alpha = 0$, (8) gives us the equation $r^2 = \frac{O(|r|^4)}{e^{-2\pi}}$. This gives us the trivial fixed point $r \approx 0$ but we are only interested in $\rho_0 > 0$. If we take $|\alpha| > 0$, from (7) we get the following equations to find the fixed point:

$$r = e^{2\pi\alpha} r + e^{2\pi\alpha}\left(\frac{1 - e^{4\alpha\pi}}{2\alpha}\right) r^3 + O(|r|^4) \tag{9}$$

$$1 = e^{2\pi\alpha} + e^{2\pi\alpha}\left(\frac{1 - e^{4\alpha\pi}}{2\alpha}\right) r^2 + O(|r|^3) \tag{10}$$

$$r^2 = \frac{1 - e^{2\pi\alpha} + O(|r|^3)}{e^{2\pi\alpha}\left(\frac{1-e^{4\alpha\pi}}{2\alpha}\right)} \tag{11}$$

For small $|r|$, the higher order terms can be discarded and the expression on the righthand side of (11) is positive for $\alpha > 0$ : While this equation has only the trivial solutions for very small $\alpha \leq 0$, there is exactly one solution for small $\alpha > 0$. After some rewriting, this point is given by

$$r \approx \sqrt{\frac{2\alpha - 2\alpha e^{2\pi\alpha}}{e^{2\pi\alpha} - e^{8\alpha\pi}}}, \quad \alpha > 0. \tag{12}$$

We can use a Taylor expansion around $\alpha = 0$, to write this fixed point as $r(\alpha) = \sqrt{\frac{2}{3}}\sqrt{\alpha} + O(\alpha^{3/2})$.

Because a positive fixed point of the Poincaré-map corresponds to a limit cycle of the system, this means that system (1) has a unique limit cycle. This limit cycle bifurcates from the origin and exists for $\alpha > 0$. A visual representation of the existence of the fixed point, is seen in the figure below.

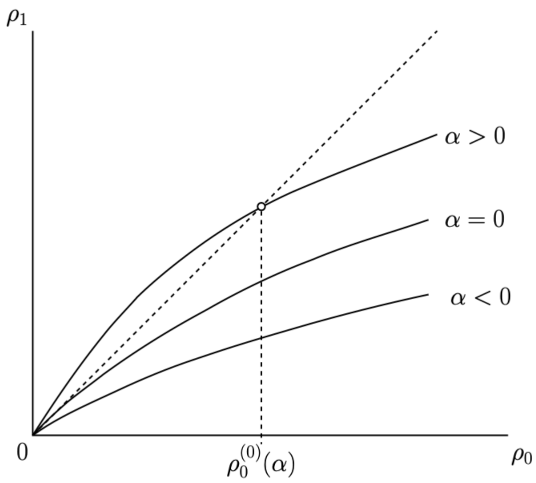


Figure 2: Visual representation of the fixed points of the Poincaré map for system (3), picture is from the book "Elements of Applied Bifurcation Theory" [2].

The existence of this fixed point is independent of any $O(|w|^4)$ terms. Therefore the higher order terms of (1) do not influence the limit cycle bifurcation in some neighborhood of $z = 0$, if we take $|\alpha|$ small enough.

At last, we will state something about the stability of the fixed point of the limit cycle. To do so we determine $|\rho_1'(r(\alpha),\alpha)|$. From (7) we see that

$$|\rho_1'(\rho_0,\alpha)| = \left| e^{2\pi\alpha} + 3e^{2\pi\alpha}\left(\frac{1-e^{4\alpha\pi}}{2\alpha}\right)\rho_0^2 + O(|\rho_0|^4)\right| \tag{13}$$

Since $r(\alpha) = \sqrt{\frac{2}{3}}\sqrt{\alpha} + O(\alpha^{3/2})$, it follows that

$$\begin{aligned} |\rho_1'(r(\alpha),\alpha)| &= \left|e^{2\pi\alpha} + e^{2\pi\alpha}(1-e^{4\alpha\pi}) + O(\alpha) + O(|r(\alpha)|^4)\right| &(14)\\ &= \left|2e^{2\pi\alpha} - e^{6\alpha\pi} + O(\alpha)\right|. &(15)\end{aligned}$$

In the last step, we have used that $r(\alpha) = O(\alpha^{1/2})$ so $O(|r(\alpha)|^4) = O(\alpha^2)$. Now from (15), we see that $\mu = \rho_1'(r(\alpha),\alpha) \in (0,1)$ if we take $\alpha$ small enough. Therefore the cycle that bifurcates from the origin is stable for small enough $\alpha$. This is again not affected by any $O(|w|^4)$ terms of (1).

# 3 Proof of uniqueness of the limit cycle of Hopf bifurcation using the Poincaré–Bendixson–Dulac theory.

In this section we will again prove the existence of an unique limit cycle of the Hopf bifurcation. This time we will use Theorem 1.15 (Bendixson-Dulac). [3]

To use the Bendixson-Dulac theory, we write down the divergence of system (3). We will use the notation $X = (\rho,\phi),\ F(X) := (\dot\rho,\dot\phi)$. The divergence is given by

$$(\mathrm{div}F)(X) = \alpha - 3\rho^2 + O(|\rho|^3). \tag{16}$$

We will now define a trapping annulus $D \in \mathbb{R}^2$ for system (3) in which the divergence is negative.

If we take $\rho$ small enough, the term $O(|\rho|^3)$ can be discarded and the divergence has the roots $\rho^* \approx \pm\sqrt{\frac{1}{3}\alpha}$. Because of our context of polar coordinates, we are only interested in the positive root and we define $\rho^* = \sqrt{\frac{1}{3}\alpha}$.

If $\rho$ gets bigger, the term $O(|\rho|^3)$ starts dominating the divergence. There are two possibilities: The first case is that there exists a smallest $\rho_3 > \rho^*$, for which the divergence has another root.
The second possibility is that such a $\rho_3$ does not exist and the divergence is negative for all $\rho > \rho^*$. Then we define $\rho_3$ as a radius bigger then $\rho^*$, for which the higher order terms can still be discarded. The value of $\rho_3$ is not important.

Now we define the annulus $D := \left\{X \in \mathbb{R}^2 :\ \dot X = F(x),\ \rho \in \left(\sqrt{\frac{1}{3}\alpha}, \rho_3.\right)\right\}$. The annulus is sketched in figure 3 below.

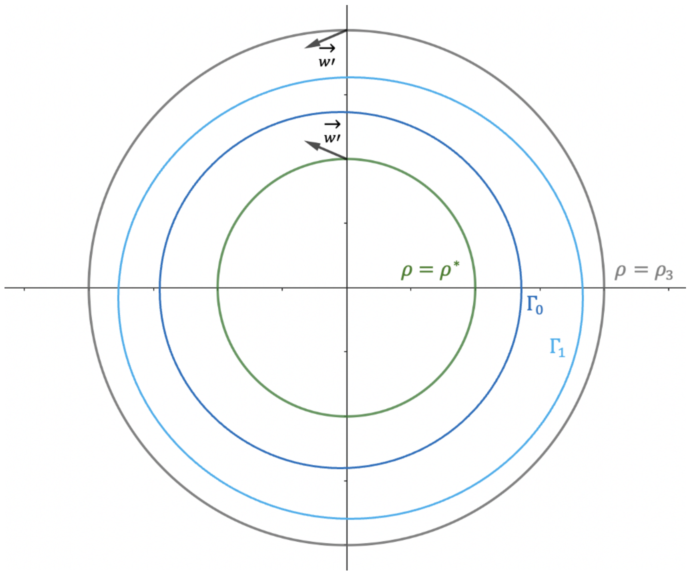


Figure 3: Visual representation of the annulus D and the limit cycles $\Gamma_0, \Gamma_1$. The flow at the borders of $D$ is sketched at the y-axis. [2].

Before we move on, we argue why $D$ is a trapping annulus for our system. Along $\Gamma_0$, we see that $\dot{\rho}|_{\rho=\rho^*} = \frac{2}{3}\sqrt{\frac{1}{3}}\alpha^{3/2} > 0$. This implies that the flow on the inner boundary of $D$ is pointing outwards. Inside $D$, the divergence is negative. We notice that (discarding higher order terms) $(\mathrm{div}F)(X) = \frac{d}{dt}\dot{\rho}$ and therefore in $D$, $\dot{\rho}$ only decreases. Furthermore we note that $\dot{\rho}|_{\rho=\sqrt{\alpha}} = 0$ while $(\mathrm{div}F)(X)|_{\rho=\sqrt{\alpha}} < 0$ and the higher order terms of (16) will not dominate for such a small $\rho = \sqrt{\alpha}$.
This means that $\rho^* < \sqrt{\alpha} < \rho_3$ and that $\dot{\rho}|_{\sqrt{\alpha}<\rho\leq\rho_3} < 0$. This implies that the flow on the outer boundary of $D$, the flow is pointing inwards.

The flow on the boundary of $D$ is sketched in figure 3. We see that the flow on the boundary of $D$ points to the interior of $D$. From this we conclude that an orbit starting in $\bar{D}$, cannot leave $\bar{D}$ and is therefore bounded.
From (1) and (3), we see that the Hopf bifurcation has only the trivial equilibrium, which lies outside of the annulus $D$. Therefore, by Theorem 2.11 (Poincaré-Bendixson) [3], an orbit starting in $\bar{D}$ must be, or approach a periodic orbit. This means that there exists at least one limit cycle $\Gamma_0$ in $\bar{D}$. This cycle is sketched in figure 3.

We want to prove that this is a unique cycle: Inside of the disc $\{X : \rho < \sqrt{\frac{1}{3}\alpha},\}$ the divergence is strictly positive and the Bendixson-Dulac theory [3] implies that there is no cycle. If a $\rho_3$ as above exists, the higher order term $O(|\rho_0|^3)$ starts dominating for $\rho > \rho_3$. In this essay we only look at small points close to $w = 0$, so we discard the area for which $\rho \geq \rho_3$.

Now assume there exists a second limit cycle $\Gamma_1$ inside $D$. (See figure 3 for a sketch.) We define the annulus $A$ which is enclosed by the cycles $\Gamma_0, \Gamma_1$, hence $\partial A = \Gamma_0 \cup \Gamma_1$.

We define $(P, Q) = F(X)$. Now by Green's theorem,

$$\oint_{\Gamma_0} Pdy - Qdx - \oint_{\Gamma_1} Pdy - Qdx = \int\int_A (\mathrm{div}F)dX < 0. \tag{17}$$

In this expression $\Gamma_0$ is traced clockwise and $\Gamma_1$ is traced counterclockwise. We can also trace the boundary of $A$ the other way around, to find

$$\oint_{\Gamma_0} Pdy - Qdx - \oint_{\Gamma_1} Pdy - Qdx = -\int\int_A (\mathrm{div}F)dX > 0. \tag{18}$$

Now both (17) and (18) lead to a contradiction because

$$\oint_{\Gamma_0} Pdy - Qdx - \oint_{\Gamma_1} Pdy - Qdx = \oint_{\Gamma_0} \langle F^\perp, dX\rangle - \oint_{\Gamma_1} \langle F^\perp, dX\rangle \equiv 0. \tag{19}$$

To conclude: In our prove we have looked at the Hopf bifurcation close to the origin. We have proved the existence of a limit cycle. The assumption of the existance of a second limit cycle lead to a contradiction. Therefore we have proven that the Hopf bifurcation only has one unique limit cycle close to the origin.

# References

[1] *topics.pdf*. Handed out for study year 2024. Univeristy Utrecht.
[2] Yuri A. Kuznetsov. "Elements of Applied Bifurcation Theory". In: Springer, 2023. Chap. Section 3.7.
[3] Yuri A. Kuznetsov. *Applied Nonlinear Dynamics*. Utrecht University, University of Twente, 2024, June 27th.