A Dual Cusp in a Sociological Model

Name: Sergio Westerhof

Student number: 2620154

Course name: Inleiding niet-lineaire dynamische systemen

Introduction and explanation of the system

Segregation is a very common phenomenon in sociological systems, it is when different groups tend to separate over time. This separation can be caused by a variety of factors, people often think that this is caused by explicit rules or for example by discrimination. Instead, most cases are more subtle, think of the combined effect of an individual's decisions which separates them over time, also known as unorganized segregation. Schelling highlighted this effect: "In some cases small incentives, almost imperceptible differentials, can lead to strikingly polarized results". This shows that even small preferences may lead to strongly segregated outcomes, which will be the focus point of this paper.

Empirical research has shown that segregation can have noticeable consequences for individuals and groups. On one hand, studies have shown that when segregation is combined with poverty it is associated with reduced exposure to positive role models, increased violence, and persistent inequalities in the society (De la Roca et al., 2014, Williamns and Collins, 2001, Cutler & Glaeser, 1997, Lewis 1966). Another study has linked residential segregation to lower education accomplishments and reduced labor market participation (Bolt 2009, Musterd, 2003). On the other hand, there are also studies who say that ethnic clustering can provide a certain social support and a sense of security (Bolt et al., 2010). These mixed effects illustrate the complexity of the segregation processes and highlights the importance of looking at how the mechanisms work in such segregation cases.

Now that we highlighted the importance of understanding such mechanisms, we will study these mechanisms through a mathematical model, which describes two interacting population groups living in the same neighborhood. The model focuses on how the population densities of these 2 particular groups will evolve over time. As an underlying basis for this paper we make use of Schelling's Bounded neighborhood model (BNM). This model provides a sociological basis for the mathematical description of segregation. In the model we look at a neighborhood which has a spatially limited area with a finite capacity. This neighborhood is inhabited by 2 different groups, denoted as X and Y, with the condition $X \geq 0, Y \geq 0$. The individuals respond to the population composition within the neighborhood and have a certain tolerance level towards the other group. Based on this tolerance level, an individual decides to remain in, enter or leave the area. A crucial aspect of the bounded neighborhood model is that segregation is not caused by external rules or policies, but from individual choices, like Schelling stated. We follow in this paper How and Hogan's approach in reformulating the bounded neighborhood model as a continuous-time dynamical system. This allows for the population densities to evolve smoothly over time while still preserving the fundamental sociological assumptions imposed by Schelling. The system of differential equations that is used to model this whole process is given below:

$$\begin{cases} \dot{x} = P(x,y) = x^2 - x^3 - xy \\ \dot{y} = Q(x,y) = \beta y^2 - \alpha\beta y^3 - xy \end{cases}$$

In the model there are 3 key mechanisms that control the behavior of it, these are derived from the paper "Dynamic models of segregation" written by Schelling and by the paper "A dynamical systems model of unorganized segregation" written by D.J. Haw and John Hogan. The mechanisms are as follows:

- The first mechanism in the model is the self-reinforcing growth of a group, represented by x^2 and βy^2. These terms in the model represent the idea that individuals are more likely to remain, or move to the neighborhood where members of their own group are already present. If the other group is not present in the environment, these terms imply positive growth. Using the interpretation of Haw and Hogan, this corresponds to the idea that a neighborhood is inherently attractive to a population when the growth is not hindered by a lack of space or by the presence of the other group.
- The second mechanism is also known as the capacity limitations and saturation effects. We see this in the terms: $-x^3$ and $-\alpha\beta y^3$. As population density increases, the available space and also the resources become limited, therefore reducing further growth of the population.
- The third and final mechanism in this model is the negative interaction term, also known as the

tolerance level, between the different groups that are in the specific neighborhood. This is represented by the term $-xy$, it represents the negative impact on growth of the presence of the other group in the neighborhood.

We have now determined what the mechanisms are in the model and it is now also important to know what the roles of α and β are in the sociological model. When looking at the $\dot{y}$ equation:

$$\dot{y} = \beta y^2 - \alpha\beta y^3 - xy$$

We can see two different parameters α, β, each parameter having a clear sociological interpretation:

- **The role of α:** The role of α can be seen when looking at the term $-\alpha\beta y^3$, it models the finite capacity of the neighborhood. α ensures when the population of Y increases, the further growth is increasingly limited. This parameter represents the only limited available space and resources. When looking in isolation (meaning $x = 0$), $\alpha > 0$ also determines directly the maximum population of Y, this means we are looking for a equilibrium where $\dot{y} = 0$:

  $$\beta y^2 - \alpha\beta y^3 = \beta y^2(1 - \alpha y) = 0 \implies y = 0 \ \vee y = \frac{1}{\alpha}$$

  This shows that α sets the maximum point of group Y for being in the neighborhood. So essentially restricting the growth of group y.

- **The role of β:** We can see that the parameter $\beta > 0$ appears in the self-reinforcing growth term, βy^2, and the limited capacity term, $-\alpha\beta y^3$. We can see on one hand if β takes on a large value, it increases the growth of the population y. On the other hand, it increases the magnitude of the limiting effect as the population approaches the point $\frac{1}{\alpha}$.

We will now look for the conditions that α, β needs to have in order for the model to be sociologically meaningful. We will show that the condition $\alpha > 0$ and $\beta > 0$ must hold. These conditions ensures that the mechanisms described above behaves like it was defined by Schelling and later on further developed by How and Hogan. To demonstrate this, we will only look at the equation of $\dot{y}$, as this is the component in which both α, β appears, we will prove that the other cases are not sociologically possible:

- We will look at the case where $\beta = 0$, if we fill this in the model and we look at the $\dot{y}$, we get therefore the following:

  $$\dot{y} = -xy$$

  If the other group is not present in the neighborhood, so the group y is in isolation (meaning x=0), the $\dot{y} = 0$. This means it will become a constant if they are in solitary for $\beta = 0$ even if there is available space to grow. This goes in contradiction to the self-reinforcing growth mechanism. Thus, $\beta = 0$ is not possible.

- We will now look at the case where $\beta < 0$. Herefore, we define $\beta = -b, b \in \mathbb{R}$ with $b > 0$, if we then look at $\dot{y}$, we get $\dot{y} = -by^2 + \alpha by^3 - xy$. If we look at this in isolation, so $x = 0$, such that only group y is present, we get:

  $$\dot{y} = -by^2 + \alpha by^3 \implies \dot{y} = by^2(-1 + \alpha y)$$

  We already know that the maximum population for Y is $\frac{1}{\alpha}$, so this means that $Y \leq \frac{1}{\alpha}$. So, when we look at these cases, we get:

  $$\dot{y} = by^2(-1 + \alpha y) \leq 0$$

  This would mean that population becomes constant or shrinks in isolation, this contradicts again the ideology the self-reinforcing growth mechanism as described above. This means that $\beta < 0$ is not possible.

- If we know look at the case where $\alpha = 0$, we get for $\dot{y}$ the following: $\dot{y} = \beta y^2 - xy$. If we look at the system in isolation, again $x = 0$, we get: $\dot{y} = \beta y^2$. This shows that $\dot{y} \geq 0$, $\forall y \geq 0$. This implies that the population grows without any bounds or is constant, this violates or the self reinforcement term or the capacity constraints of the model. This means that α can not be 0.
- We finally look at the case if $\alpha < 0$, we define $\alpha = -a,\ a \in \mathbb{R}$ with $a > 0$, if we rewrite $\dot{y}$ we get $\dot{y} = \beta y^2 + a\beta y^3 - xy$. If we look at this in isolation we get $\dot{y} = \beta y^2 + a\beta y^3 \geq 0,\ \forall y \geq 0$. This means that the system explodes again, contradicting the conditions of the bounded neighborhood model or the system becomes constant contradicting the mechanism of the self reinforcing term. Therefore α can not achieve a negative value.

We have shown that the sociological model only works for $\alpha, \beta > 0$. We are now going to argue that this model is a segregation model. Firstly, we must explain what a segregation model is, we define it as a mathematical framework which represents the core mechanisms that were identified by Schelling, that individual choices can lead to a large separation between groups. When we take a look at our system we can see, as mentioned before, 3 different mechanisms. The quadratic self growth terms, being $x^2, \beta y^2$, model the inclination of an agent to remain or move to the neighborhood where there are more member of his own group. If we look at the cubic capacity limitation term,$-x^3, -\alpha\beta y^3$, it imposes that there is a certain finite capacity, therefore limiting unbounded growth. When looking at the last term $-xy$, it captures the negative effect of the presence of the other group. Combining all these 3 mechanisms it produces a system where it can lead to complete domination of one group but under certain conditions it can produce mixed neighborhoods. This means it translates the individual preferences and the local interaction into the model, highlighting that small changes in individual preferences can lead to different outcomes, meaning it captures segregation. Therefore the system is a segregation model.

Equilibrium points of the system

We will now compute the equilibria of this sociological model and also determine their stability. We have derived previously that the model is only suitable for $\alpha, \beta > 0$.
Firstly, we are going to compute the equilibria, this means that we are interested where $\dot{x} = 0$ and $\dot{y} = 0$, so:

$$\dot{x} = x^2 - x^3 - xy = 0$$
$$\dot{y} = \beta y^2 - \alpha\beta y^3 - xy = 0$$

If we solve this further we get:

$$\dot{x} = x^2 - x^3 - xy = x(x - x^2 - y) = 0 \implies x = 0 \quad \vee \quad x - x^2 = y$$
$$\dot{y} = \beta y^2 - \alpha\beta y^3 - xy = y(\beta y - \alpha\beta y^2 - x) = 0 \implies y = 0 \quad \vee \quad x = \beta y - \alpha\beta y^2$$

We have now derived all the possible conditions for having an equilibrium. We will now look at all the possibilities:

- **Case $x = 0$:** We look at the case when $x = 0$, we have stated previously the following for $\dot{y}$:

  $$\dot{y} = 0 \implies x = \beta y - \alpha\beta y^2 \quad \vee \quad y = 0$$

  If $x = 0$ we can rewrite the results of $\dot{y} = 0$ to the following:

  $$0 = \beta y - \alpha\beta y^2 \implies 0 = \beta y(1 - \alpha y) \quad \vee \quad y = 0$$

  We are interested in what happens when $x = 0$ so we look at:

  $$\beta y(1 - \alpha y) = 0 \implies \beta y = 0 \quad \vee \quad 1 - \alpha y = 0$$

  We find the following options:

  $$y = 0 \quad \vee \quad y = \frac{1}{\alpha}$$

  This means that when $x = 0$, which was an equilibrium condition, there are two y coordinates that satisfy $\dot{y} = 0$, namely, $y = 0$ or $y = \frac{1}{\alpha}$. So we found two equilibrium points $(0, 0), (0, \frac{1}{\alpha})$.

- **Case $y = 0$:** We look at the case when $y = 0$, we have stated previously that for $\dot{x}$:

  $$\dot{x} = 0 \implies x = 0 \quad \vee \quad y = x - x^2$$

  If $y = 0$ then we can rewrite $\dot{x}$ to the following:

  $$x - x^2 = 0 \quad \vee \quad x = 0,$$

  we can rewrite the first equation as follows:

  $$x - x^2 = x(1 - x) = 0 \implies x = 0 \quad \vee \quad x = 1$$

  Thus, for $y = 0$, which was an equilibrium condition, there are two x conditions that satisfy $\dot{x} = 0$, namely, $x = 0$ or $x = 1$. So we found two equilibrium points for the case when $y = 0$: $(0, 0), (1, 0)$

- **Case $x \neq 0 \wedge y \neq 0$:** Now we are interested if there exists an equilibrium when both x and y are not 0. In the beginning of this paper we introduced x and y as population densities, this means that they never can be zero in the real world, so in this case we are interested if there exists an equilibrium when $x > 0, y > 0$. This means that we are looking at the following system, because we neglect $x = 0$ and $y = 0$ as a possibility for an equilibrium point:

  $$\begin{cases} y = x - x^2 \\ x = \beta y - \alpha\beta y^2 \end{cases}$$

If we substitute $y = x - x^2$ in $x = \beta y - \alpha\beta y^2$, we may get certain equilibrium points:

$$\begin{aligned} x =& \beta(x - x^2) - \alpha\beta(x - x^2)^2 \\ &\beta x - x - \beta x^2 - \alpha\beta(x^2 + x^4 - 2x^3) = 0 \\ &x(\beta - \beta x - \alpha\beta x - \alpha\beta x^3 + 2\alpha\beta x^2 - 1) = 0 \end{aligned}$$

We have stated before that we are looking where x, y are greater than zero so we can look at the following:

$$\beta - \beta x - \alpha\beta x - \alpha\beta x^3 + 2\alpha\beta x^2 - 1 = 0$$

If we rewrite the equation we get:

$$-\alpha\beta x^3 + 2\alpha\beta x^2 - \beta x(1 + \alpha) + (\beta - 1) = 0$$

We already know that $\alpha, \beta > 0$, meaning that $\alpha\beta \neq 0$, this means we can multiply the equation by $-\frac{1}{\alpha\beta}$, which gives us:

$$x^3 - 2x^2 + \frac{\beta x(1 + \alpha)}{\beta\alpha} - \frac{\beta - 1}{\alpha\beta} = 0$$

$$x^3 - 2x^2 + \frac{(\alpha + 1)}{\alpha}x + \frac{1 - \beta}{\alpha\beta} = 0$$

Next, we define the coefficients as follows:

$$a_3 = 1, \quad a_2 = -2, \quad a_1 = \frac{\alpha + 1}{\alpha}, \quad a_0 = \frac{1 - \beta}{\alpha\beta}$$

We are interested in the solutions of this cubic equation, in order to compute these solutions we must first calculate the discriminant, to determine how many real solutions this cubic equation has. Here, we will use the definition of the discriminant for a cubic equation, because $a_3 = 1$ we get the following formula for the discriminant:

$$D = a_2^2 a_1^2 - 4a_1^3 - 4a_2^3 a_0 - 27a_0^2 + 18a_2 a_1 a_0$$

If we fill in the coefficients we determined earlier we can derive a formula for the discriminant in terms of α, β, we get:

$$D = \frac{4(\alpha + 1)^2}{\alpha^2} - \frac{4(\alpha + 1)^3}{\alpha^3} + \frac{32(1 - \beta)}{\alpha\beta} - \frac{27(1 - \beta)^2}{(\alpha\beta)^2} - \frac{36(\alpha + 1)(1 - \beta)}{\alpha^2\beta}$$

We look at a common denominator, in our case it will be $\alpha^3\beta^2$. We will rewrite the whole equation with the denominator being $\alpha^3\beta^2$. We will do so by looking at each term individually and later combining it:

- **Term $\frac{4(1+\alpha)^2}{\alpha^2}$:** If we rewrite this, with the knowledge $(1 + \alpha)^2 = \alpha^2 + 2\alpha + 1$, we get the following:

$$\frac{4(1 + \alpha)^2}{\alpha^2} = \frac{4(1 + \alpha)^2\alpha\beta^2}{\alpha^3\beta^2} = \frac{(4\alpha^3 + 8\alpha^2 + 4\alpha)\beta^2}{\alpha^3\beta^2}$$

- **Term $-\frac{4(1+\alpha)^3}{\alpha^3}$:** If we rewrite this and with the knowledge that: $(1 + \alpha)^3 = \alpha^3 + 3\alpha^2 + 3\alpha + 1$ then we get:

$$-\frac{4(1 + \alpha)^3}{\alpha^3} = -\frac{4(1 + \alpha)^3\beta^2}{\alpha^3\beta^2} = \frac{(-4\alpha^3 - 12\alpha^2 - 12\alpha - 4)\beta^2}{\alpha^3\beta^2}$$

– **Term** $\frac{32(1-\beta)}{\alpha\beta}$**:** If we rewrite this we get the following:

$$\frac{32(1-\beta)}{\alpha\beta} = \frac{32(1-\beta)\alpha^2\beta}{\alpha^3\beta^2} = \frac{32\alpha^2\beta - 32\alpha^2\beta^2}{\alpha^3\beta^2}$$

– **Term** $-\frac{27(1-\beta)^2}{\alpha^2\beta^2}$**:** If we rewrite this, with knowledge that $(1-\beta)^2 = 1 - 2\beta + \beta^2$, then we can rewrite it accordingly:

$$-\frac{27(1-\beta)^2}{\alpha^2\beta^2} = -\frac{27\alpha(1-\beta)^2}{\alpha^3\beta^2} = \frac{-27\alpha + 54\alpha\beta - 27\alpha\beta^2}{\alpha^3\beta^2}$$

– **Term** $-\frac{36(1+\alpha)(1-\beta)}{\alpha^2\beta}$**:** If we rewrite this with again the knowledge that $(1+\alpha)(1-\beta) = 1 - \beta + \alpha - \alpha\beta$, then it can be rewritten as:

$$-\frac{36(1+\alpha)(1-\beta)}{\alpha^2\beta} = -\frac{36\alpha\beta(1+\alpha)(1-\beta)}{\alpha^3\beta^2} = \frac{-36\alpha\beta + 36\alpha\beta^2 - 36\alpha^2\beta + 36\alpha^2\beta^2}{\alpha^3\beta^2}$$

If we now combine every term in the original equation but now rewritten so that the denominator is $\alpha^3\beta^2$ we get:

$$\begin{aligned} D = \frac{(4\alpha^3 + 8\alpha^2 + 4\alpha)\beta^2}{\alpha^3\beta^2} + \frac{(-4\alpha^3 - 12\alpha^2 - 12\alpha - 4)\beta^2}{\alpha^3\beta^2} + \frac{32\alpha^2\beta - 32\alpha^2\beta^2}{\alpha^3\beta^2} \\ + \frac{-27\alpha + 54\alpha\beta - 27\alpha\beta^2}{\alpha^3\beta^2} + \frac{-36\alpha\beta + 36\alpha\beta^2 - 36\alpha^2\beta + 36\alpha^2\beta^2}{\alpha^3\beta^2} \end{aligned} \tag{1}$$

We can rewrite the formula in only one fraction:

$\frac{(4\alpha^3+8\alpha^2+4\alpha)\beta^2+(-4\alpha^3-12\alpha^2-12\alpha-4)\beta^2+32\alpha^2\beta-32\alpha^2\beta^2-27\alpha+54\alpha\beta-27\alpha\beta^2-36\alpha\beta+36\alpha\beta^2-36\alpha^2\beta+36\alpha^2\beta^2}{\alpha^3\beta^2}$

We will rewrite this fraction to a much simpler version, we will first group all the terms:

– **Terms containing** β^2**:** We can group the terms of β^2 accordingly:

$$\beta^2(4\alpha^3 + 8\alpha^2 + 4\alpha - 4\alpha^3 - 12\alpha^2 - 12\alpha - 4 - 32\alpha^2 - 27\alpha + 36\alpha + 36\alpha^2)$$

If we re-order this we see the following:

$$\begin{aligned} &= \beta^2(\alpha^3(4-4) + \alpha^2(8 - 12 - 32 + 36) + \alpha(4 - 12 - 27 + 36) - 4) \\ &= \beta^2(\alpha - 4) \end{aligned}$$

– **Terms containing** β**:** We group the terms of β accordingly:

$$\beta(32\alpha^2 + 54\alpha - 36\alpha - 36\alpha^2) = \beta(-4\alpha^2 + 18\alpha)$$

– **Terms containing constants:** The constant terms were just simply -27α.

If we rewrite the discriminant using the terms that were derived above we obtain:

$$D = \frac{\beta^2(\alpha - 4) + \beta(-4\alpha^2 + 18\alpha) - 27\alpha}{\alpha^3\beta^2}$$

If we factor out a minus sign from the numerator it gives us:

$$D = -\frac{\beta^2(4 - \alpha) + \beta(4\alpha^2 - 18\alpha) + 27\alpha}{\alpha^3\beta^2}$$

For a cubic equation the sign of the discriminant determines the number of real solutions. If $D > 0$ the equation has 3 distinct real roots, while when $D < 0$ it has only one real root. Because the discriminant has an overall minus sign, the sign of D is opposite to the standard cubic discriminant. We therefore define a modified discriminant:

$$D_{new} = -D$$

By defining it as follows, the cubic equation has 3 real roots if $D_{new} < 0$ and has one real root if $D_{new} > 0$ We follow the same normalization that was used by Haw and Hogan, so we rescale the discriminant by a factor of $\frac{1}{108}$. This doesn't affect its sign because $\frac{1}{108} > 0$, this gives us:

$$D_{new} = \frac{(4-\alpha)\beta^2 + (4\alpha^2 - 18\alpha)\beta + 27\alpha}{108\alpha^3\beta^2}$$

We are interested in determining for which conditions $D_{new} < 0$ and $D_{new} > 0$, because these conditions give control over the number of real equilibria in the system. We will first calculate the conditions where $D_{new} = 0$, we have previously determined the discriminant as:

$$D_{new} = \frac{(4-\alpha)\beta^2 + (4\alpha^2 - 18\alpha)\beta + 27\alpha}{108\alpha^3\beta^2}$$

Since $\alpha, \beta > 0$, the denominator $108\alpha^3\beta^2$ is strictly positive. Therefore the sign of D_{new} only depends on the numerator:

$$(4-\alpha)\beta^2 + (4\alpha^2 - 18\alpha)\beta + 27\alpha$$

For the case to determine where the discriminant is equal to 0 we will look at where:

$$(4-\alpha)\beta^2 + (4\alpha^2 - 18\alpha)\beta + 27\alpha = 0$$

By using the quadratic formula, we can determine for which conditions it separates the region of the two different areas of the equilibria points. For clarity we will use D_1 for the discriminant of the quadratic formula. We will use the general quadratic formula with the form: $A\beta^2 + B\beta + C = 0$, which results to:

$$D_1 = B^2 - 4AC, \quad \beta_{\pm} = \frac{-B \pm \sqrt{D}}{2A}$$

We will first calculate the D_1:

$$\begin{aligned} D_1 &= (4\alpha^2 - 18\alpha)^2 - 4 \cdot (27\alpha) \cdot (4-\alpha) \\ &= 16\alpha^4 + 324\alpha^2 - 144\alpha^3 - 432\alpha + 108\alpha^2 \\ &= 16\alpha^4 - 144\alpha^3 + 432\alpha^2 - 432\alpha \\ &= 16\alpha(\alpha^3 - 9\alpha^2 + 27\alpha - 27) \end{aligned}$$

We want to rewrite the equation inside the brackets, we will do so accordingly:

$$\begin{aligned} \alpha^3 - 9\alpha^2 + 27\alpha - 27 &= \alpha^3 - 6\alpha^2 + 9\alpha - 3\alpha^2 + 18\alpha - 27 \\ &= (\alpha - 3)(\alpha^2 - 6\alpha + 9) = (\alpha - 3)(\alpha - 3)(\alpha - 3) \\ &= (\alpha - 3)^3 \end{aligned}$$

So we can rewrite our discriminant D_1:

$$D_1 = 16\alpha(\alpha - 3)^3$$

We have derived an equation for D_1 so we can now determine $\beta_{\pm}$:

$$\beta_{\pm} = \frac{-(4\alpha^2 - 18\alpha) \pm \sqrt{16\alpha(\alpha-3)^3}}{2(4-\alpha)}$$

We can factor the 2 out of the equation:

$$\beta_{\pm} = \frac{9\alpha - 2\alpha^2 \pm 2\sqrt{\alpha(\alpha-3)^3}}{4-\alpha}$$

We have now a condition for $D_{new} = 0$ which is a line that gives the separation of the region where there are 3 real solutions and where there is only 1 real solution. A requirement for this is that $\alpha \geq 3$, because then $\sqrt{\alpha(\alpha-3)^3}$ becomes real. We will now look at what happen at each region of α when looking at this equation, so that we can later present in which region the $D_{new} > 0$ takes place and where $D_{new} < 0$ takes place. We will look at 3 different cases for α:

- **Case 1** $\alpha \in [3,4)$**:** We can look at both β_-, β_+ what happens when $\alpha \in [3,4)$. We can see that in this case the denominator is always positive. When looking at the numerator we can study it numerically and conclude that for $\alpha \in [3,4)$, it will attain a positive value, so for this case both bounds are completely fine.
- **Case 2** $\alpha = 4$**:** When $\alpha = 4$, the denominator will be 0 which can not be the case because then it will diverge meaning there is for $\alpha = 4$, a vertical asymptote. This means that $\alpha \neq 4$.
- **Case 3** $\alpha > 4$**:** When looking at the case $\alpha > 4$ we come up with something interesting. If we study it numerically we can see that for $\alpha > 4$ β_+ will attain a negative value because the denominator will be negative but the nominator will attain a positive value. As stated many times in this paper, the condition $\beta > 0$ should be held. Meaning that when α will grow further than 4 β_+ will cross into the negative β-region and therefore is excluded by the condition $\beta > 0$, this means that β_+ will never cross the vertical asymptote. When looking at β_- when $\alpha > 4$, we see that when studying it numerically both numerator and denominator becomes negative attaining a positive value. This means that β_- will grow further and as seen in the figure underneath it creates a certain region.

We can present these regions of $\beta_{\pm}$ into a 2D system in terms of β and α:

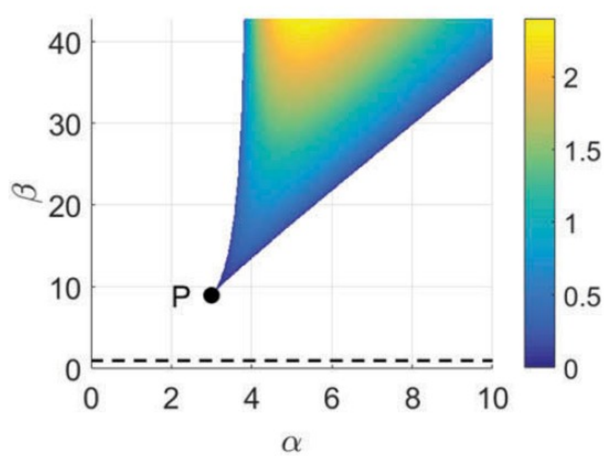


Figure 1: Region cases (Haw, D.J., & Hogan,S. J., 2018)

We have now determined the regions that are created. We can can see that when the discriminant D_{new} is negative we are operating in the colored region that is shown in Figure 2. So as a result the system will have then 3 mixed equilibria with $x > 0, y > 0$. We can now also conclude that the other region belongs to when the discriminant $D_{new} > 0$, meaning there exists only one mixed equilibria for $x > 0, y > 0$.

Stability of the equilibrium points

Now we are going to look at the stability of each of these equilibria points. Firstly it is handy to compute the Jacobian that we will use to determine the stability in each equilibrium point. We know that the Jacobian is defined as follows:

$$J(x,y) = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix}$$

We will calculate each component of the matrix:

$$\frac{\partial \dot{x}}{\partial x} = 2x - 3x^2 - y$$

$$\frac{\partial \dot{x}}{\partial y} = -x$$

$$\frac{\partial \dot{y}}{\partial x} = -y$$

$$\frac{\partial \dot{y}}{\partial y} = 2\beta y - 3\alpha\beta y^2 - x$$

This gives us the Jacobian in the following way:

$$J(x,y) = \begin{pmatrix} 2x - 3x^2 - y & -x \\ -y & 2\beta y - 3\alpha\beta y^2 - x \end{pmatrix}$$

We will now look at the stability of each point, we will do so accordingly:

- **Equilibrium point** $(0,0)$**:** We will first compute the Jacobian by filling in the our equilibrium point in the Jacobian, we get the following:

  $$J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

  We now try to determine the the eigenvalues of the system in order to try and say something about the stability of the system. We will use the following formula that do so in the following way:

  $$det(J(0,0) - \lambda I) = det \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

  If we work this out then:

  $$\lambda^2 = 0 \implies \lambda_{1,2} = 0$$

  We know that when both λ are equal to 0, then we are looking at a degenerate case.
  Normally we should look at the higher order terms and determine the stability of it, but in this case we consider this not necessary because the equilibrium point is in $(0,0)$. This means that it corresponds to an empty neighborhood, which we define as not interesting. Instead, we will focus on the remaining equilibria.

- **Equilibrium point** $(0, \frac{1}{\alpha})$**:** We will first fill in the Jacobian by filling the equilibrium point:

  $$J(0, \frac{1}{\alpha}) = \begin{pmatrix} -\frac{1}{\alpha} & 0 \\ -\frac{1}{\alpha} & \frac{2\beta}{\alpha} - \frac{3\alpha\beta}{\alpha^2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha} & 0 \\ -\frac{1}{\alpha} & -\frac{\beta}{\alpha} \end{pmatrix}$$

  We are now going to compute the eigenvalues for the point $(0, \frac{1}{\alpha})$. We will do so, in the same way as done above:

  $$det(J(0, \frac{1}{\alpha}) - \lambda I) = det \begin{pmatrix} -\frac{1}{\alpha} - \lambda & 0 \\ -\frac{1}{\alpha} & -\frac{\beta}{\alpha} - \lambda \end{pmatrix}$$

If we work this out:

$$= (-\frac{1}{\alpha} - \lambda)(-\frac{\beta}{\alpha} - \lambda) = 0 \implies \lambda_1 = -\frac{1}{\alpha} \quad \vee \quad \lambda_2 = -\frac{\beta}{\alpha}$$

We have stated previously that for our model the following condition should be held: $\alpha, \beta > 0$, this means that both eigenvalues $\lambda_1 < 0,\ \lambda_2 < 0$. So we know that both our eigenvalues are real and negative for the equilibrium point $(0, \frac{1}{\alpha})$, this means that the point is a stable node.

- **Equilibrium point** $(1, 0)$**:** Here again, we will first fill in the Jacobian to later on try to compute the eigenvalues of the system:

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

  Now we will calculate the eigenvalues:

$$det(J(1,0) - \lambda I) = det \begin{pmatrix} -1-\lambda & -1 \\ 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1-\lambda) = 0 \implies \lambda_{1,2} = -1$$

  We can conclude with this that both eigenvalues are negative and real meaning that the point $(1, 0)$ is also a stable node.

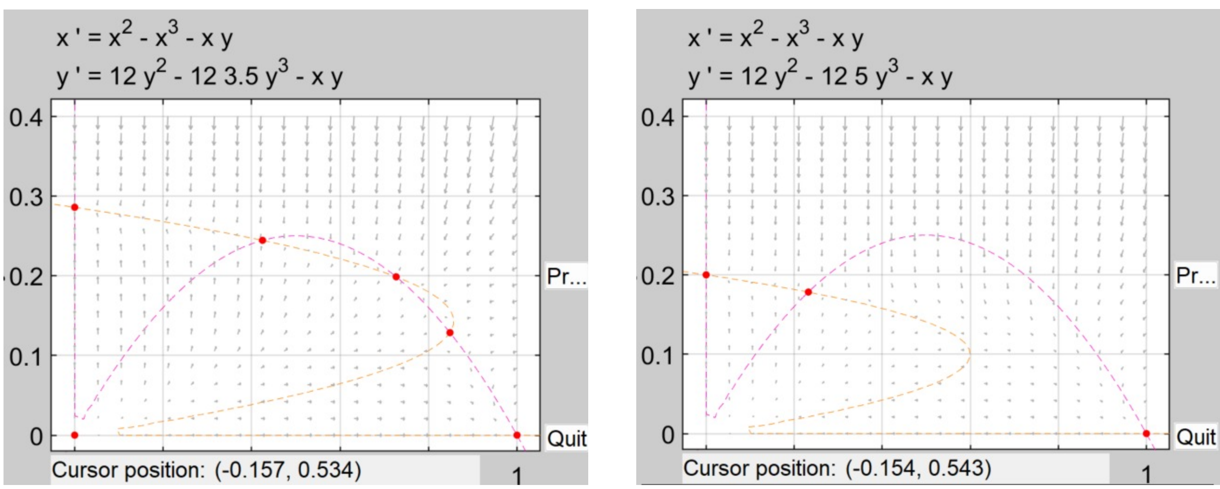
- **Equilibria points when** $x \neq 0$ **and** $y \neq 0$**:** We will now look at the stability of the mixed equilibria that was given by the cubic equation that we presented in the section before. We can not easily determine algebraically the stability of this equilibrium condition that we have derived previously. That is why we will study it numerically, here we will use also the numerical analysis done by How and Hogan. According to numerical analysis and inspection they showed that when inside the region created by β_+ and β_- one equilibrium point attains a stable point and that the other 2 real solutions are saddle points. If the points lie outside the region, meaning they are outside the bounds of $\beta_\pm$ they showed that there is only a single real solution which will behave like a saddle for these values of (α, β). We will present in the next section phase portraits that were done by using MATLAB. In the phase portrait we could indeed see that the region that is between β_+ and β_- attains 3 mixed equilibria points; two saddle points and one stable node. We also looked at the phase portrait that was made for outside of this region we could conclude that there was only one mixed equilibrium point which was a saddle point. As long as the parameters α and β remain within the same region defined by the lines of D_{new} the stability of these equilibria will not change. This confirms the numerical analysis done by How and Hogan.

Phase portraits and bifurcation in the system

We will now look at the behavior of the system as the parameters of α and β changes. We are particularly interested in the changes regarding the number of stability of the mixed equilibria. We know from the previous section that the mixed equilibria are given by the solutions of the cubic equation that we defined earlier:

$$\beta_{\pm} = \frac{9\alpha - 2\alpha^2 \pm 2\sqrt{\alpha(\alpha-3)^3}}{4-\alpha} \quad , \alpha \geq 3$$

We can look what happens when the determinant is smaller than 0, meaning we are looking at the case where there are 3 real equilibria points. We can also look outside the region where we have only one equilibrium, due to the fact that the determinant is greater than 0.
To understand the importance of these regions, it is not enough to look at the number of the equilibria points and the stability of it but also look at the trajectories they have when time evolves. Phase portraits provide here a clear visualization of what happens for a fixed α, β. These phase portraits can also show us whether the system converges to integration or segregation of the system, when looking at different initial population compositions. We will now look at the phase portraits in each of these 2 regions, we will explain them accordingly:



(a) When $D_{new} < 0$, so the system has 3 mixed equilibria, for the parameters $\alpha = 3.5,\ \beta = 12$

(b) When $D_{new} > 0$, meaning the system has 1 mixed equilibrium, for the paramters $\alpha = 5,\ \beta = 12$

Figure 2: Comparison of the two discriminant cases

When looking at the phase portraits it reveals to us the long-term behavior of the system in both regions. In the first figure (a) we can clearly see that we are inside the region as shown before. We have 3 mixed equilibria, being one stable node and the are 2 are saddle points. The stable node attracts the nearby trajectories. This point represents the stable mixed population for the long term. When we look at the behavior of the saddle points in the system, it shows us that they have a great influence where they direct the flow to. They can push the flow towards the stable mixed equilibrium, but if the population composition is different they can move it outwards towards the fully dominated neighborhood, represented by $(1, 0), (0, \frac{1}{\alpha})$. So depending on what the population composition is, these saddle points will redirect the flow towards the mixed stable equilibria or to the dominated neighborhood equilibria. If we look at when we are only dealing with one mixed equilibrium, shown in figure (b), we have only one equilibrium which is a saddle point. This is a unstable equilibrium point, representing an equilibrium point for a short period of time but when there are some small changes in the population composition it redirects the flow towards the two dominated neighborhood points $(1, 0), (0, \frac{1}{\alpha})$.

The phase portraits of above describe the system for a certain fixed parameters. If we want to understand how these behaviors change as the parameters change, it is good to study the bifurcation happening in the system. The bifurcation diagram is shown below:

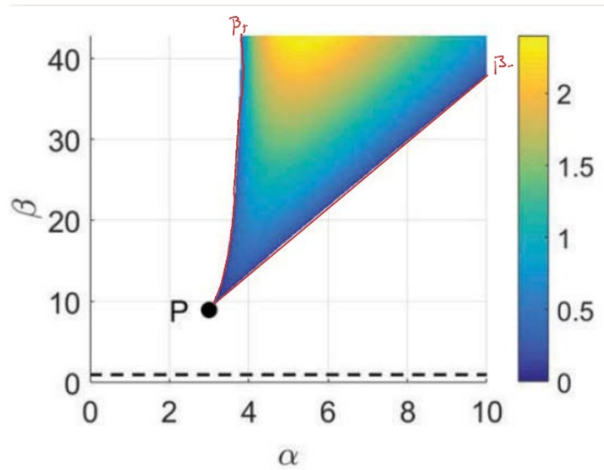
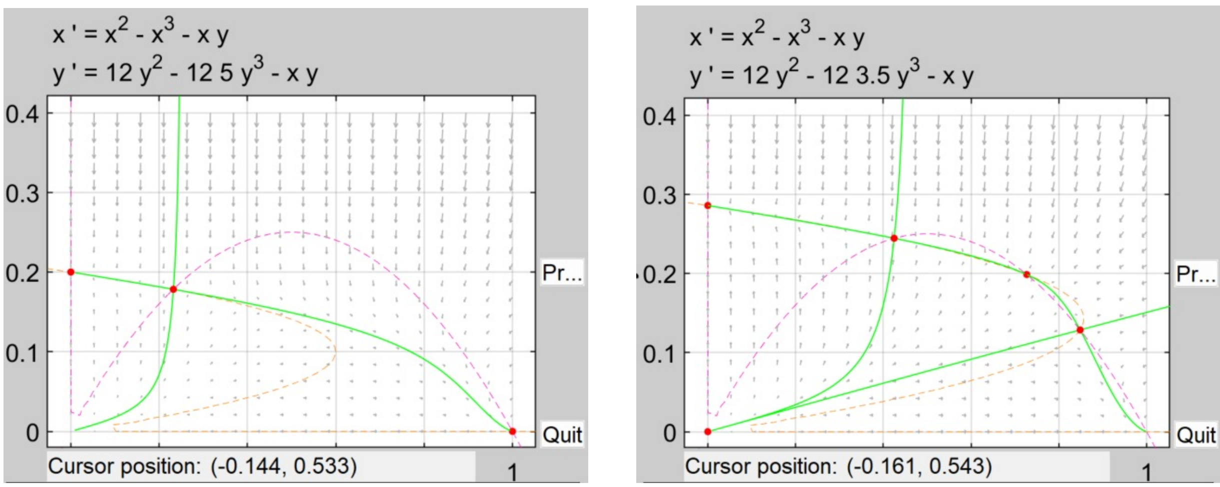


Figure 3: Bifurication diagram

We can state clearly that there are some bifurcations happening in this model. We have derived previously that when trespassing the line $\beta_{\pm}$ we are moving from having three mixed equilibria to one mixed equilibrium or vice versa. When we look at the figure above we have highlighted these two lines with the color red. We can see them both as bifurcation lines, to be more specific each of these lines are fold bifurcations, where a stable equilibrium and a saddle equilibrium collide and annihilate each other or the other way around. This changes the number of mixed equilibria in the system, showing that if we want to have a mixed stable equilibrium the parameters must be in a certain region.

We can define the point P, seen in figure 3, as the point where the 2 fold bifurcations lines meet, here a dual cusp bifurcation occurs. This point separates qualitatively different regions, small changes in parameters around this point can lead to sudden changes in the number and stability of the equilibria.

We have seen that the phase portraits show the overall direction of the trajectories, but they do not explain well what happens when looking at long term outcomes if we look at the nearby initial conditions. This means that when we look at a small population in the neighborhood we can determine towards which equilibrium point it will go to. To understand this sensitive behavior better, it is necessary to look at the invariant manifolds that are associated with the saddle equilibria points. These determine the boundaries between different dynamical behaviors. Presented below in green are the manifolds:



(a) Manifolds in the case of 1 mixed equilibrium

(b) Manifolds in the case of 3 mixed equilibria

Figure 4: Comparison of manifolds for different numbers of mixed equilibria

The presence of the saddle equilibria and with their manifolds show that the the long-term sociological outcome does not solely depend on α and β but also on the initial population composition. With the manifolds being present we can look at which equilibrium small initial population mix will go to. We can see that the manifolds act as boundaries determining whether the system evolves towards a stable mixed equilibrium or towards a complete segregation. This means that especially when there are not a lot of people in the neighborhood small changes near these manifolds lines can compete change the outcomes later on.

Conclusion

So to conclude, we looked at the continuous-time system of unorganized segregation which was based on the Bounded Neighborhood Model with the help of the reformulation given by Haw and Hogan. We have analyzed the equilibria and stability of the system. When looking at the mixed equilibrium points we have showed that small changes in the parameters (α, β) can lead to different long-term outcomes, varying from stable mixed neighborhood to complete segregation. An interesting point was the presence of the dual cusp bifurcation, because from there the parameter spaces separates the regions where there exist one mixed equilibrium point or three mixed equilibrium points. Here we set the lines of $\beta_{\pm}$ as our fold bifurcation lines, meaning the line where the equilibrium points changes. Later on, we also looked at the phase portraits and the manifolds to show that not only the parameter values are important but also the composition of the initial population plays a crucial role in determining the final state of the model. This shows that small individual preferences and small changes can have strong impacts on segregation patterns. This highlights the sensitivity and the complexity of these social systems.

References

The references that were used to write this paper are as follows:

- Bolt, G. (2009). Combating residential segregation of ethnic minorities in European cities. *Journal of Housing and the Built Environment, 24*(4), 397–405. https://doi.org/10.1007/s10901-009-9163-z
- Bolt, G., Özüekren, A. S., & Phillips, D. (2010). Linking integration and residential segregation. *Journal of Ethnic and Migration Studies, 36*(2), 169–186. https://doi.org/10.1080/13691830903387238
- Cutler, D. M., & Glaeser, E. L. (1997). Are ghettos good or bad? *The Quarterly Journal of Economics, 112*(3), 827–872. https://doi.org/10.1162/003355397555361
- De la Roca, J., Ellen, I. G., & O'Regan, K. M. (2014). Race and neighborhoods in the 21st century: What does segregation mean today? *Regional Science and Urban Economics, 47*, 138–151. https://doi.org/10.1016/j.regsciurbeco.2013.09.006
- Hanßmann, H., & Momin, A. (2024). Dynamical systems of self-organized segregation. *Journal of Mathematical Sociology, 48*(3), 279–310. https://doi.org/10.1080/0022250X.2023.2267762
- Haw, D. J., & Hogan, S. J. (2018). A dynamical systems model of unorganized segregation. *Journal of Mathematical Sociology, 42*(3), 113–127. https://doi.org/10.1080/0022250X.2018.1430603
- Kuznetsov, Y. A. (2013). *Elements of applied bifurcation theory* (3rd ed.). Springer. https://doi.org/10.1007/978-1-4757-3978-7
- Lewis, O. (1966). The culture of poverty. *Scientific American, 215*(4), 19–25. https://doi.org/10.1038/scientificamerica 19
- Musterd, S. (2003). Segregation and integration: A contested relationship. *Journal of Ethnic and Migration Studies, 29*(4), 623–641. https://doi.org/10.1080/1369183032000123422
- Schelling, T. C. (1971). Dynamic models of segregation. *Journal of Mathematical Sociology, 1*(2), 143–186. https://doi.org/10.1080/0022250X.1971.9989794
- Williams, D. R., & Collins, C. (2001). Racial residential segregation: A fundamental cause of racial disparities in health. *Public Health Reports, 116*, 404–416. https://doi.org/10.1016/S0033-3549(04)50068-7