Lorenz systems Project

Niels van Dijk

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1 Introduction

The Lorenz system is a system of differential equations, which was created by Edward Lorenz in 1963 to model atmospheric convection. This system is known to be chaotic: small changes in initial values or parameters can yield widely different outcomes for the system as a whole.

We are going to study this system, by taking a look at it physical interpretations, and studying the parameter dependent stability of fixed points of the system.

1.1 What is the Lorenz system?

An often considered thought experiment is that of the butterfly effect. This says that actions that might seem small at first can have major consequences, like a butterfly flapping its wings could eventually effect a storm.

Chaotic systems are similar. In these systems of differential equations, slightly different initial conditions of the system can have vastly different solutions, making it unpredictable. Of course, for a specific initial value, you could still calculate a solution, allowing you to predict the future, but this will not say much about what happens close by.

Edward Lorenz summarizes chaotic systems as follows:

Chaos: When the present determines the future but the approximate present does not approximately determine the future.

One example of a chaotic system, and perhaps the most famous one, is the Lorenz system:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = x(\rho - z) - y, \\ \dot{z} = xy - \beta z, \end{cases} \tag{1}$$

We assume the parameters σ, ρ and β to be greater than 0.

This system was created by the mathematician and meteorologist Edward Lorenz together with Ellen Fetter, who was responsible for numerical calculations and figures, and Margaret Hamilton who helped in the initial, numerical computations leading up to the finding of the system, to model atmospheric convection [3].
In the system, x stands for the rate of convection in the atmosphere, which is air raising in altitude because of differences in air temperature.
y stands for horizontal temperature variation and z is the vertical temperature variation.

The parameter σ is called the Prandtl number, and ρ is called the Rayleigh number.

2 Analysis of the system

2.1 Finding the equilibria

We are going to find the parameter-dependent equilibria of the Lorenz system. We assume all parameters are greater than 0, and solve

$$\begin{cases} \sigma(y-x)=0 \\ x(\rho-z)-y=0 \\ xy-\beta z=0 \end{cases}$$

Since $\sigma > 0$, it follows that we must have $y - x = 0,\ x = y$. We substitute this into the third equation to find that we must have $x^2 - \beta z = 0, z = \frac{x^2}{\beta}$. With this information, we find that the second equation becomes:

$$x(\rho - \frac{x^2}{\beta}) - x = \ x(\rho - \frac{x^2}{\beta} - 1) = 0$$

So $x = 0$ or $\rho - \frac{x^2}{\beta} - 1 = 0$, i.e. $x = \pm\sqrt{\beta(\rho-1)}$.

Thus the equilibria of the Lorenz system are

$$(x,y,z) = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1) \text{ and } (x,y,z) = (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1),$$

$$\text{or } (x,y,z) = (0,0,0).$$

We observe that, when $\rho < 1$, there can only be one equilibrium, the origin. When $\rho = 1$, the first two equilibria coincide, so there are two equilibria, and when $\rho > 1$, all equilibria exist.
Because the amount of equilibria changes as we let ρ vary, a bifurcation occurs when $\rho = 1$.

Interestingly, the equilibria are not dependent on σ, only on the parameters ρ and β.

From now on, we refer to the equilibria $(x,y,z) = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$ by $Q_\pm$.

2.2 Stability of the equilibria, dependent on the parameters

Now that we have found the equilibria of the Lorenz system, we are going to determine their stability. When we write the system as a vector field $F(x, y, z)$, we find that the total derivative is equal to

$$DF(x,y,z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}$$

We substitute the equilibria, one by one, to find what their (parameter-dependent) stability is.

2.2.1 (x,y,z)=(0,0,0)

We first observe the origin. The total derivative at (0,0,0) is given by:

$$DF(0,0,0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}, \text{ so } |DF(0,0,0) - \lambda I| = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ \rho & -1-\lambda & 0 \\ 0 & 0 & -\beta-\lambda \end{vmatrix}$$

$$= -(\lambda+\beta)(\lambda^2 + \lambda(\sigma+1) + \sigma - \rho\sigma).$$

We can immediately see that one of the eigenvalues is $\lambda_1 = -\beta$. To find the other two eigenvalues, we need to solve $\lambda^2 + \lambda(\sigma+1) + \sigma - \rho\sigma = 0$. Using the quadratic formula, we find that the solutions are

$$\lambda_{2,3} = \frac{1}{2}(-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)})$$

Suppose $\rho < 1$, then we see that all eigenvalues are negative. This is because $1 - \rho \in (0, 1)$, and for those values $(\sigma + 1^2) > 4\sigma(1-\rho)$. So $\sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}$ is real, and when $\rho < 1$ it is smaller than $\sigma + 1$, so in this case λ_2 and λ_3 are both negative.

Thus, for $\rho < 1$, we only have the equilibrium $(x, y, z) = (0, 0, 0)$, which has three real negative eigenvalues. So it is a stable node. [1]

Now, let $\rho = 1$. In this case, the eigenvalues of the origin are: $\lambda_1 = -\beta, \lambda_2 = 0, \lambda_3 = -(\sigma + 1)$. This is not a stable node anymore. A bifurcation occurs here.

Let $\rho > 1$. We can evaluate the eigenvalues to be:

$$\lambda_1 = -\beta,\ \lambda_{2,3} = \frac{1}{2}(-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)})$$

We now have $1 - \rho < 0$, so $\sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)} > \sigma + 1$.
Thus, the eigenvalues are $\lambda_1 = -\beta < 0$, $\lambda_2 = \frac{1}{2}(-(\sigma+1) + \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}) > 0$ and $\lambda_3 = \frac{1}{2}(-(\sigma+1) - \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}) < 0$.

Since we have two eigenvalues < 0 and one eigenvalue > 0, it follows that the origin is an attracting saddle when $\rho > 1$, independent of the values of β and σ.

2.2.2 Global stability of the origin for $\rho < 1$

When $\rho < 1$, we have already seen that the origin is a stable equilibrium point, and it's the only equilibrium point. Using a Lyapunov function, we are able to prove that the origin is globally stable: all solutions of the system tend to the origin.

Definition 2.1. A Lyapunov function for a system

$$\begin{cases} f : \mathbb{R}^n \to \mathbb{R}^n \\ \dot{y} = f(y) \end{cases}$$

with an equilibrium point at $y = 0$ is a scalar function $g : \mathbb{R}^n \to \mathbb{R}$ that is continuous, strictly positive for $y \neq 0$, has continuous first derivatives and has time derivative non-positive.

Theorem 2.2. *Let the Lyapunov function g be globally positive definite, the equilibrium $y = 0$ be isolate, the time derivative of the Lyapunov function be globally negative definite and satisfy*

$$\|x\| \to \infty \implies g(x) \to \infty.$$

Then the equilibrium $y = 0$ is globally asymptotically stable.

Take the function $g(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$. Since $\sigma > 0$, it follows that this function is strictly positive for $y \neq 0$. It is also continuous, and the time derivative along the flow of g is given by:

$$\frac{\partial g}{\partial t} = 2xy - 2x^2 + 2xy(\rho - z) - 2y^2 + 2xyz - 2\beta z^2 = -2x^2 + 2xy + 2\rho xy - 2y^2 - 2\beta z^2.$$

It might not be directly clear that this is ≤ 0, but that will follow when we rewrite it as:

$$= -2\left(x - \frac{\rho + 1}{2}y\right)^2 - 2\left(1 - \left(\frac{\rho + 1}{2}\right)^2\right)y^2 - 2\beta z^2$$

Now, the first part has to be negative because the square result in a positive number. Since $\frac{\rho+1}{2} < 1$ it follows that the second term is negative as well. Finally, $-\beta z^2$ is also negative, thus $\frac{\partial g}{\partial t} \leq 0$: g is a Lyapunov function.

To fulfill the conditions of the theorem, we need to show that g is globally positive definite, which requires us to additionaly prove that $g(x, y, z) = 0$ only when $(x, y, z) = (0, 0, 0)$, and that its time derivative is globally negative definite: so $\frac{\partial g}{\partial t} = 0$ only when $(x, y, z) = (0, 0, 0)$.

First, it follows because of sums of squares and knowing $\sigma > 0$ that $g(x, y, z) = 0$ only when $(x, y, z) = (0, 0, 0)$.
Secondly, the time derivative of g has all terms negative, so we just need every term to be zero: this means $-2\beta z^2 = 0$, so $z = 0$. Similarly, $-2(1 - (\frac{\rho+1}{2})^2)y^2 = 0$, so $y = 0$ because the inside of the parentheses cannot be equal to 0 (that requires $\rho = 1$, which we do not consider). Finally, we now need $-2(x - \frac{\rho+1}{2}y)^2 = 0$ i.e. $-2x^2 = 0$, it follows that $x = 0$ as well. Thus $(x, y, z) = (0, 0, 0)$.

We satisfy all the conditions of Theorem 2.2, so for $0 < \rho < 1$ the origin, which is the only equilibrium of the system, is globally asymptotically stable.

Thus, all solutions of the system go towards the origin. This is the most boring case of the system, everything is predictable. This changes drastically when we let ρ be bigger.

2.2.3 Stability of $Q_\pm$

Now that we have observed the stability of the origin for $\rho < 1$, we take a look at the other two equilibria, which only appear once $\rho > 1$.

The total derivative at $Q_\pm$ is given by the Jacobian matrix

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} & \pm\sqrt{\beta(\rho-1)} & -\beta \end{pmatrix}$$

We want to calculate the eigenvalues of this matrix, which we can do applying the Laplace expansion on the determinant

$$\begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} & \pm\sqrt{\beta(\rho-1)} & -\beta-\lambda \end{vmatrix}$$

$$= -(\sigma+\lambda)\begin{vmatrix} -1-\lambda & \mp\sqrt{\beta(\rho-1)} \\ \pm\sqrt{(\beta(\rho-1))} & -\beta-\lambda \end{vmatrix} - \sigma\begin{vmatrix} 1 & \mp\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} & -\beta-\lambda \end{vmatrix}$$

$$\begin{aligned} &= -(\sigma+\lambda)(\lambda^2+(\beta+1)\lambda+\beta\rho) - \sigma(-\lambda+\beta(\rho-2)) \\ &- \lambda^3 - (\sigma+\beta+1)\lambda^2 - (\sigma\beta+\sigma+\beta\rho)\lambda - \sigma\beta\rho + \lambda\sigma - \beta\sigma(\rho-2) \\ &= -\lambda^3 - (\sigma+\beta+1)\lambda^2 - \beta(\sigma+\rho)\lambda - 2\beta\sigma(\rho-1). \end{aligned}$$

So the eigenvalues of $Q_\pm$ can be determined by finding the roots of the characteristic equation

$$\lambda^3 + (\sigma+\beta+1)\lambda^2 + \beta(\sigma+\rho)\lambda + 2\beta\sigma(\rho-1) = 0$$

We would like to find the eigenvalues, which are the roots of the characteristic equation above. Since this is a polynomial of degree 3 however, that becomes difficult because of the complexity of the cubic equation.

2.3 Hopf bifurcation existence.

To find stability of the equilibria $Q_\pm$, we only need to determine the sign of the eigenvalues, which we can do using the Routh-Hurwitz stability criterion. We will first define the Routh-Hurwitz matrix before we introduce the criterion.

Definition 2.3. (Routh-Hurwitz matrix) let $P(s)$ be the polynomial

$$P(s) = a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0.$$

Then the Routh-Hurwitz matrix has the following structure:

$$\begin{pmatrix} a_n; & a_{n-2}; & a_{n-4}; & \dots \\ a_{n-1}; & a_{n-3}; & a_{n-5}; & \dots \\ \frac{a_1a_2-a_0a_3}{a_1}; & \frac{a_1a_4-a_0a_5}{a_1}; & \dots & \dots \\ \frac{\frac{a_1a_2-a_0a_3}{a_1}a_3-\frac{a_1a_4-a_0a_5}{a_1}a_1}{\frac{a_1a_2-a_0a_3}{a_1}}; & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

In general, after the first two rows, the entry r_{kj} at the intersection of the k-th row and j-th column is given by the fraction

$$r_kj = \frac{r_{k-1,1}r_{k-2,j+1} - r_{k-2,1}r_{k-1,j+1}}{r_{k-1,1}}.$$

We repeat the algorithm until the number of rows in the matrix is equal to $(n+1)$.

Using this matrix gives us the following useful theorem:

Theorem 2.4. *(Routh-Hurwitz stability criterion) Let $P(s)$ be the polynomial*

$$P(s) = a_ns^n + a_{n-1}s^{n-1} + ... + a_1s + a_0.$$

Then the roots of $P(s)$ have real part < 0 if and only if all the entries in the first column of the Routh-Hurwitz matrix have the same sign.

This criterion follows from the Routh-Hurwitz Theorem. A proof can be found at [6]. The following Lemma also follows from the Routh-Hurwitz Theorem.

Lemma 2.5. *Let $P(s)$ be the polynomial*

$$P(s) = a_ns^n + a_{n-1}s^{n-1} + ... + a_1s + a_0.$$

The amount of sign changes in the first column of the Routh-Hurwitz matrix is the amount of roots with real parts nonnegative.

This criterion can give us exactly what we want: A way to know when the eigenvalues of $Q_\pm$, which are the roots of the characteristic polynomial, all have negative real part, so the equilibria are stable.

We had the characteristic polynomial

$$P(\lambda) = \lambda^3 + (\sigma+\beta+1)\lambda^2 + \beta(\sigma+\rho)\lambda + 2\beta\sigma(\rho-1).$$

So we can construct its Routh-Hurwitz matrix to be:

$$\begin{pmatrix} 1 & \beta(\sigma+\rho \\ \sigma+\beta+1 & 2\beta\sigma(\rho-1) \\ (\sigma+\beta+1)\beta(\sigma+\rho) - 2\beta\sigma(\rho-1) & 0 \\ 2\beta\sigma(\rho-1) & 0 \end{pmatrix}$$

Therefore, we obtain the following requirements for $Q_\pm$ to be stable:

$$\sigma + \beta + 1 > 0,\ (\sigma + \beta + 1)\beta(\sigma + \rho) - 2\beta\sigma(\rho - 1) > 0,\ 2\beta\sigma(\rho - 1) > 0.$$

The first and third requirements following immediately, since we assume $\sigma, \beta, \rho > 0$, and the equilibria we are studying only exist for $\rho > 1$.

From now on, we assume that $\sigma > \beta + 1$, because this occurs most often in nature and this allows us to derive a condition for stability.

We are now going to work out the second requirement. We can write it out into

$$(\sigma + \beta + 1)\beta\sigma + \rho\beta(\sigma + \beta + 1) + 2\beta\sigma + \rho(-2\beta\sigma) > 0$$

We want to isolate ρ, since the existence of equilibria is solely dependent on it. This we rewrite the in equality as follows

$$\rho\beta(-\sigma + \beta + 1) > -\sigma\beta(\sigma - \beta - 3).$$

By using our assumption that $\sigma > \beta + 1$

$$\rho < \frac{-\sigma\beta(\sigma + \beta + 3)}{\beta(-\sigma + \beta + 1)} = \sigma\frac{\sigma + \beta + 3}{\sigma - \beta - 1}$$

Thus we obtain a criteria on ρ for the stability of $Q_\pm$.

With this in hand, we can show that a Hopf bifurcation occurs when $\rho = \sigma\frac{\sigma+\beta+3}{\sigma-\beta-1}$. We call this value ρ_H from now on.

We use the following Lemma from [1]:

Lemma 2.6. *Let $P(s)$ be the third-order polynomial*

$$P(s) = \lambda^3 + p\lambda^2 + q\lambda + r$$

and let $R = pq - r$. Then the characteristic equation has at least one zero root on the surface $r = 0$, and a pair of imaginary eigenvalues on the surface $(R = 0, q > 0)$.

In our case $R = (\sigma + \beta + 1)\beta(\sigma + \rho) - 2\beta\sigma(\rho - 1)$, which we know to be equal to zero exactly when $\rho = \rho_H$. Since $q = 2\beta\sigma(\rho - 1) > 0$ when $\rho > 1$, it follows that $Q_\pm$ has a pair of purely imaginary eigenvalues when $\rho = \rho_H$. The first column of the Routh-Hurwitz has two sign changes, so the remaining eigenvalue, λ_3 has negative real part.
In fact, λ_3 must be real, because for real polynomials, the conjugate of a root is also a root, but if $\mathrm{im}(\lambda_3) \neq 0$, there is no eigenvalue left to be the conjugate.

To conclude: When $\rho = \rho_H$, the equilibria $Q_\pm$ have a pair of purely imaginary eigenvalues λ_1, λ_2, and one real eigenvalue $\lambda_3 < 0$. Thus a Hopf bifurcation occurs here.

2.4 We found Hopf bifurcations, are they supercritical or subcritical?

Now that we have shown a Hopf bifurcation occurs in the system for $\rho = \rho_H$, we are going to determine whether this bifurcation is subcritical or supercritical, thus proving non-degeneracy. To do this, we will need to calculate the Lyapunov coefficient.

Firstly, we transform the Lorenz system into one equation. We write $y = \frac{\dot{x}}{\sigma} + x$, and substitute this into the $\dot{y}$ equation to obtain:

$$\frac{\ddot{x}}{\sigma} + \dot{x} = x(\rho - z - 1) - \frac{\dot{x}}{\sigma}.$$

Next, we incorporate the $\dot{z}$ equation

$$z = \frac{-\ddot{x}}{\sigma x} - \frac{\dot{x}}{x} + \rho - \frac{\dot{x}}{\sigma x} - 1.$$

We can substitute this into the $\dot{z}$ equation, to obtain

$$\dddot{x} + (\sigma + \beta + 1)\ddot{x} + \beta(\sigma + 1)\dot{x} + \beta\sigma(1 - \rho)x = \frac{(1+\sigma)\dot{x}^2}{x} + \frac{\ddot{x}\dot{x}}{x} - x^2\dot{x} - \sigma x^3.$$

This is a single equation encapsulating the Lorenz system. To calculate the Lyapunov coefficient, we need to translate it such that Q_+ or Q_- is at the origin. We substitute $\xi = x - x_0$, where $x_0 = \pm\sqrt{\beta(\rho - 1)}$ for Q_+ and Q_- respectively, it will not end up mattering which one we take. The equation becomes

$$\dddot{\xi} + (\sigma + \beta + 1)\ddot{\xi} + (\beta(1 + \sigma) + x_0^2)\dot{\xi} + (\beta\sigma(1 - \rho) + 3\sigma x_0^2)\xi$$
$$= -3\sigma x_0 \xi^2 - 2x_0\xi\dot{\xi} + \frac{1+\sigma}{x_0}\dot{\xi}^2 + \frac{1}{x_0}\dot{\xi}\ddot{\xi} - \sigma\xi^3 - \xi^2\dot{\xi} - \frac{1+\sigma}{x_0^2}\xi\dot{\xi}^2 - \frac{1}{x_0^2}\xi\dot{\xi}\ddot{\xi} + ...$$

where we have taken the taylor approximation around zero of $\frac{1}{\xi + x_0}$ as:

$$\frac{1}{\xi + x_0} \sim \frac{1}{x_0} - \frac{\xi}{x_0^2} + ...$$

we do not consider higher order terms, since those are not necessary to calculate the Lyapunov coefficient.

Using an algorithm from [1] (page 877-879) it follows that the Lyapunov coefficient is given by:

$$L_1 = \beta(p^3 q(p^2 + q)(p^2 + 4q)(\sigma - \beta - 1))^{-1}B,$$

where

$$B = 9\sigma^4 + (20 - 18\beta)\sigma^3 + (20\beta^2 + 2\beta + 10)\sigma^2 - (2\beta^3 - 12\beta^2 - 10\beta + 4)\sigma - \beta^4 - 6\beta^3 - 12\beta^2 - 10\beta - 3.$$

Unfortunately, I do not know what p and q are here... But they are positive, and it turns out that by substitution of $\sigma = \sigma_* + b + 1$ in the B-equation makes all coefficients become positive.

Thus the Lyapunov coefficient L_1 is positive. By definition, this means that the Hopf bifurcations that occur at $Q_\pm$ for $\rho = \rho_H$ are subcritical.

3 The Lorenz attractor

We are now going to observe the sequence of global bifurcations of the Lorenz system that leads to the creation of the Lorenz attractor.

First, we define attractors.

Definition 3.1. Let f be the flow of an system. An attractor is a subset A of the phase space characterized by the following conditions:

1 If $a \in A$, then so is $f(t, a)$ for all $t > 0$.

2 There exists a neighborhood of A, called $B(A)$, such that for all $b \in B(A)$ and for all open neighborhoods N of A, there is a $T > 0$ such that $f(t, b) \in N$ for all $t > T$.

3 There is no non-empty subset of A with the previous two properties.

Some examples of 2D attractors are:
stable equilibria, since there is no movement happening inside it, the first condition holds. When we look close enough, everything around it is attracted towards it, so the second condition holds. We can not find a subset with these two properties, because A is a single point.

Another example is a stable periodic orbit in $\mathbb{R}^2$. If a on the periodic orbit A, then $f(t, a)$ is on A for all $t > 0$, because of periodicity.
Since the periodic orbit is stable, solutions close to it are attracted towards it, from both the inside and outside.
Finally, we cannot break the periodic orbit and retain the previous two properties.

Now that we have defined attractors, we are going to study global bifurcations of the Lorenz system, and we eventually see an attractor appear.

We fix $\sigma = 10$ and $\beta = \frac{8}{3}$, and let ρ vary between 10 and 30.

When $\rho \approx 10$, we see the following:

As we can see in figure 1, the two unstable seperatrices move towards the stable equilibria $Q_{\pm}$, and spiral around. When we increase ρ, this changes.

Around $\rho_t \approx 13.93$, the unstable seperatrices don't just move around Q_+ or Q_+, but instead they are attracted to the origin, which is a saddle. They form two homoclinic orbits, which can be seen in Figure 2:

This is where a homoclinic bifurcation occurs. As a result of this bifurcation, periodic orbits come into existence as a result of the destruction of the homoclinic orbits as we let ρ increase.

This can be observed in Figure 3, where we have shown a diagram of the system with ρ greater than $\rho_t \approx 13.93$, but smaller than $\rho_H \approx 24.74$, where the Hopf bifurcations occur.

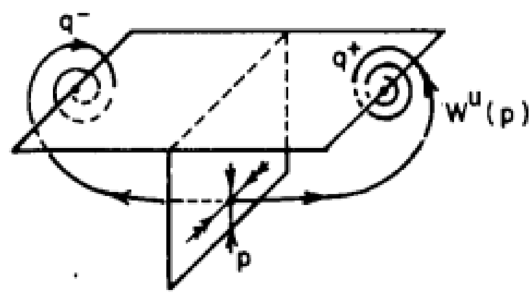

Figure 1: Flow of the Lorenz system for $\rho \approx 10,\ \sigma = 10,\ \beta = \frac{8}{3}$. Image from [2]

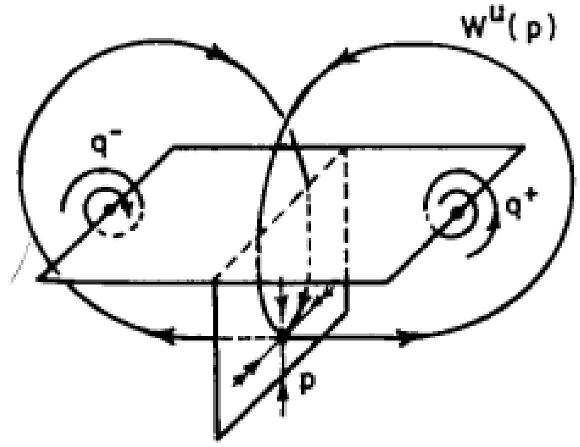

Figure 2: Flow of the Lorenz system for $\rho = \rho_t \approx 13.93,\ \sigma = 10,\ \beta = \frac{8}{3}$. Image from [2]

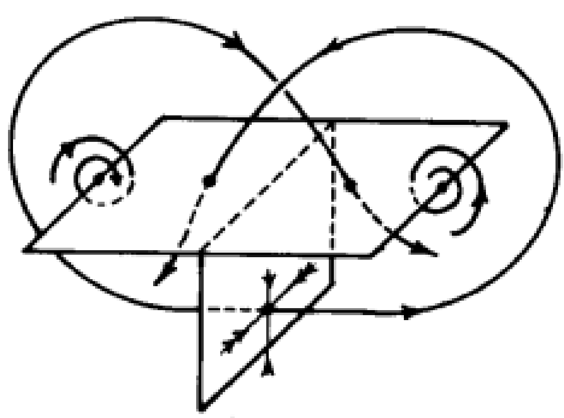

Figure 3: Flow of the Lorenz system for $\rho > \rho_t \approx 13.93$ but smaller than $\rho_H \approx 24.74,\ \sigma = 10,\ \beta = \frac{8}{3}$. Image from [2]

The seperatrices now move around the other stable equilibrium $Q_\pm$ then they did for $\rho < \rho_t$ (So the seperatrix that went around Q_+ now goes around Q_- and vice versa). This can be interpreted as the effect of the stable equilibria becoming weaker. This is also visible in the following diagram: (Figure 4)

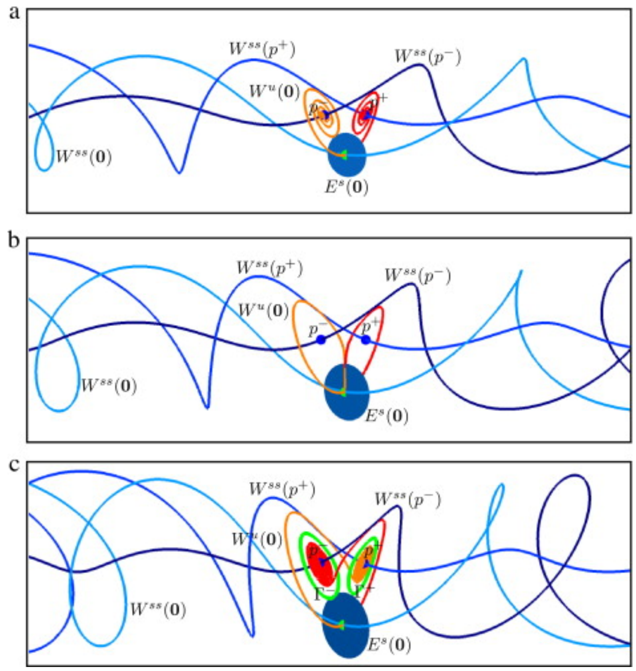

Figure 4: A diagram showing three states of the system when varying ρ: The first is before the homoclinic bifurcation, so $\rho < \rho_t$. The second is during the bifurcation, so $\rho = \rho_t$. The third state is after the homoclinic bifurcation, where periodic orbits have come into existence. Image from [4]

We can now see that there exist periodic orbits after the homoclinic bifurcation, and the seperatrices are attracted to the opposite equilibrium point now.

In the range of $\rho_t < \rho < \rho_H$, transient chaos occurs. This means that solutions with very close initial conditions can still have wildly different behavior, but they eventually settle down. In our case this means that solutions eventually stay close to one of the equilibria $Q_\pm$, and do not jump to the other anymore.

Finally, when ρ reaches ρ_H, the hopf bifurcations take place. Now, $Q_\pm$ lose their stability, and the periodic orbits, which were produced by the homoclinic bifurcation, are destroyed. What remains, is the Lorenz attractor. This looks as follows: (Figure 5)

Solutions outside the attractor all go towards it, but once they enter the attractor, chaos occurs. Solutions that have initial conditions very close to each other have completely different behaviour relative to each other. This is called a strange attractor.

The Lorenz system also looks like the wings of a butterfly. This is where the name 'butterfly effect' comes from.

A weather model like the Lorenz system showing chaos explains why weather forecasts get more unreliable the further in the future we look. We could calculate the weather conditions next year for example, but because the real initial conditions might be slightly different, the predictions become completely unreliable because of the chaos.

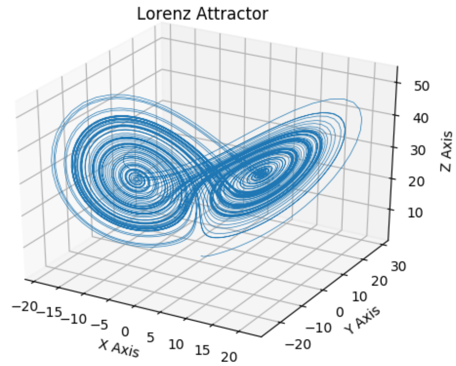

Figure 5: The Lorenz attractor, shown with parameter values $\rho = 28$, $\sigma = 10$, $\beta = \frac{8}{2}$. Figure from [5]

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