Period 3 implies chaos

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1. Introduction

Chaos theory studies how simple deterministic systems can produce highly complex and unpredictable behavior. A central theme in one-dimensional discrete dynamical systems is the relationship between periodic orbits and chaotic dynamics. In particular, the appearance of periodic points of certain orders often signals the onset of complicated long-term behavior. One of the most striking results in this area is the theorem of Li and Yorke, commonly summarized by the phrase "period three implies chaos." This theorem establishes that the existence of a single periodic orbit of minimal period three forces the existence of periodic orbits of every possible period, revealing an inherent richness in the system's dynamics.

The first part of this essay focuses on stating some notation and known definition and results as an refresher. Then we move on to proving the Li–Yorke theorem for continuous mappings of the unit interval. We show that if a continuous function $F : [0, 1] \to [0, 1]$ possesses a cycle of minimal period three, then it must also possess cycles of minimal period n for all integers $n \geq 1$. This result highlights how a seemingly simple periodic structure guarantees an infinite hierarchy of periodic behaviors. After that we discuss the theorem and its proof.

In the second part, we give a short introduction into the logistic map $f(x) = \alpha x(1 - x)$, a classical model in population dynamics and chaos theory. We also give examples of its different regimes and bifurcations. Also we apply the Li&Yorke theorem to ideas to the logistic map. We analyze the behavior of the third iterate of the logistic map and demonstrate that at the critical parameter value $\alpha_0 = 1 + 2\sqrt{2}$, a fold (saddle-node) bifurcation occurs. This bifurcation gives rise to a pair of period-three cycles, one stable and one unstable, as the parameter α increases. Together, these results illustrate both the abstract theory and its concrete manifestation in a fundamental nonlinear system, reinforcing the deep connection between periodicity and chaos.

Throughout this paper, we assume familiarity with the standard local bifurcations of one-dimensional maps, including fold (saddle-node), transcritical, and period-doubling bifurcations. For precise definitions and normal forms, see for example Y. A. Kuznetsov. Elements of Applied Bifurcation Theory [5] and Steven Strogatz. Nonlinear Dynamics and Chaos [10].

2. Priminilary definitions and results

This section briefly reviews standard definitions and clarifies notation for the sake of clarity in the subsequent sections. We first define an **iterative map**. Let $F : X \to X$ be a function defined on a set X, then the sequence defined by repeated application of F on $x_0 \in X$

$$x_{n+1} = F(x_n)$$

is called an *iterative map* or *discrete dynamical system*. The n-th iterate of F on x_0 is denoted by $F^n(x_0) = x_n$ where

$$F^n(x_0) = \underbrace{F \circ F \circ \cdots \circ F(x_0)}_{n \text{ times}}$$

Note that $F^0(x_0) = x_0$ is the identity map. For a initial point/condition $x_0 \in X$ we define the **orbit** of x_0 under F as the sequence

$$\mathcal{O}(x_0) = \{x_0, F(x_0), F^2(x_0), F^3(x_0), \dots\}$$

We follow through with the some definitions

Definition 1 *Let $n \in \mathbb{N}$. We say that p is a periodic point of period n if $p \in X$ and $p = F^n(p)$ and $p \neq F^k(p)$ for $1 \leq k < n$. For such p we call $\mathcal{O}(p)$ a periodic orbit. Note also that every point in $\mathcal{O}(p)$ then has period n. [7]*

Definition 2 *Let $F : X \to X$ be a continuous function, we call $p \in X$ a fixed point of F if $F(p) = p$. We call it a stable fixed point if $|F'(p)| < 1$, unstable if $|F'(p)| > 1$ and neutral if $|F'(p)| = 1$.*

Now we move on to one of the key principles that is used the proof of the Li&Yorke theorem.

Theorem 3 ***: Intermediate Value Theorem***
Let $F : [a, b] \to \mathbb{R}$ be a continuous function. If $y \in \mathbb{R}$ such that

$$F(a) \leq y \leq F(b) \quad or \quad f(b) \leq y \leq f(a),$$

then there exists a point $c \in [a, b]$ such that $f(c) = y$.

Corollary 4 *For a continuous function $F : [a, b] \to \mathbb{R}$ it holds that $[F(a), F(B)] \subset F[a, b]$*

We move to the actual Li&Yorke theorem and prove it.

3. Li&Yorke Theorem and proof

Theorem 5 (Li& Yorke) *Let J be an interval and let $F : J \to J$ be continuous. Assume there are points $a, b, c, d \in J$ for which $F(a) = b$, $F^2(a) = c$, and $F^3(a) = d$ and either (Case 1)*

$$d \leq a < b < c$$

or (Case 2)

$$d \geq a > b > c.$$

$T_1 :$ *Then for every $k \in \mathbb{N}$ there exists a periodic point in J with period k.*
$T_2 :$ *there exists an uncountable set S in J (containing no periodic points) which satisfies the following conditions:*
(A) For every $p, q \in S$ with $p \neq q$.

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0$$

and

$$\liminf_{n \to \infty} |F^n(p) - F^n(q)| = 0$$

(B) For every $p \in S$ and periodic point $q \in J$.

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0$$

Remark 6 *Note that if $d = a$ then $F^3(a) = a$. Thus a is a periodic point of period three. So the hypothesis is satisfied by the existence of a periodic point of period three. [7]*

Here, T_1 represents the main theorem and T_2 is Li&Yorke their definition of chaos. Roughly speaking, Li and Yorke called a map chaotic if it contains an uncountable set of points whose orbits behave in an irregular and contradictory way. For any two distinct points in this set, their orbits are required to be both arbitrarily close at some times and significantly separated at other times. In other words, the orbits repeatedly come close together and then drift far apart, without settling into any predictable pattern. This behavior excludes convergence to fixed points or periodic orbits and captures a strong form of long-term unpredictability. Li and Yorke's key insight was to show that the existence of a single periodic orbit of period three guarantees the existence of such an irregular set. Thus, period three does not merely imply many periodic orbits—it implies chaos in this precise topological sense. This definition later became known as Li–Yorke chaos, and although stronger notions of chaos have since been introduced, it remains one of the most influential and conceptually clear definitions in one-dimensional dynamics. We will leave to the reader if to read more about chaos if they are interested in it. A great book about chaos is [2], here chaos is defined in a more modern way. Anyway, we found it crucial that one has a bit of intuitive understanding of what chaos is about as defined in [7].
However we will not prove T_2. If one is interested in the proof of T_2 one can find a very detailed version in [3].

Proof [T_1]:
To proof the main part of the Li&Yorke theorem we need to have certain foundations to further expand on. Since these foundations are essential, we see them as part of the whole proof. Therefor we will state them as lemmas which necessary for the proof of the actual theorem. In addition, we will also prove these lemmas.

Lemma 7 *Let $G : I \to \mathbb{R}$ be continuous, where I is an interval. For any compact interval $I_1 \subseteq G(I)$ there is a compact interval $Q \subseteq I$ such that $G(Q) = I_1$.*

Proof [Lemma 7]
Let $I_1 \subseteq G(I)$ and I_1 is compact. Then because of continuity $\exists p, q \in I$ such that $I_1 = [G(p), G(q)]$. If $p < q$, let

$$r = \max\{x \in [p, q] \mid G(x) = G(p)\}$$

and let

$$s = \min\{x \in [r, q] \mid G(x) = G(q)\}.$$

Then by the intermediate value theorem $G[r, s] = I_1$. If $p > q$, let

$$r = \max\{x \in [q, p] \mid G(x) = G(q)\}$$

and let

$$s = \min\{x \in [r, p] \mid G(x) = G(p)\}.$$

Then by the intermediate value theorem $G[r, s] = I_1$. Note that in both cases $[r, s]$ is compact because it is a closed interval. Therefore there is a compact interval $Q \subseteq I$ such that $G(Q) = I_1$. [3] ∎

It is important to note that since G is not necessarily one to one then $I_1 \subseteq G[p, q]$ by corollary 4, but perhaps not $G[p, q] \subseteq I_1$.

Lemma 8 *Let $F : J \to J$ be continuous and let $\{I_n\}_{n=0}^{\infty}$ be a sequence of compact intervals with $I_n \subset J$ and $I_{n+1} \subseteq F(I_n)$ for all n. Then there is a sequence of compact intervals Q_n such that $Q_{n+1} \subset Q_n \subset I_0$ and $F^n(Q_n) = I_n$ for $n \in \mathbb{N}$. Define $Q = \bigcap_{n=0}^{\infty} Q_n$ then for any $x \in Q$, we have $F^n(x) \in F^n(Q_n) = I_n$ for all n.*

Remark 9 *Note that because Q_n is compact, it is therefore not empty because it at least contains one point. Furthermore, we have $Q_{n+1} \subset Q_n$ for all $n \in \mathbb{N}$ and so $Q = \bigcap_{n=0}^{\infty} Q_n \neq \emptyset$. Also note that if F, G are continuous functions then $F \circ G$ is also continuous.*

Proof [Lemma 8] we prove this lemma via using lemma 7 inductively. For the base case, define Q_0 such that $F^0(Q_0) = I_0$, i.e $Q_0 = I_0$. By lemma 7 $\exists Q_1 \subset Q_0$ such that $F(Q_1) = I_1 \subset F(I_0)$. Now suppose $\exists Q_{n+1} \subset Q_n$ such that $F^{n+1}(Q_{n+1}) = I_{n+1} \subset F(I_n)$. Then by lemma 7 $\exists Q_{n+2} \subset Q_{n+1}$ such that $F^{n+2}(Q_{n+2}) = I_{n+2} \subset F(I_{n+1})$. This completes the induction.
Since $Q \neq \emptyset$, then for any $x \in Q$ it follows that $x \in Q_n$ for all $n \in \mathbb{N}$. Hence trivially $F^n(x) \in F^n(Q_n) = I_n$ for all $n \in \mathbb{N}$. [3] ■

Lemma 10 *Let $G : J \to \mathbb{R}$ be continuous. Let $I \subset J$ be a compact interval. Assume $I \subset G(I)$. Then there is a point $p \in I$ such that $G(p) = p$.*

Proof [Lemma 10] Let $I = [\beta_0, \beta_1]$. Since $I \subseteq G(I)$ we can choose $\alpha_0, \alpha_1 \in I$ such that $G(\alpha_0) = \beta_0$ and $G(\alpha_1) = \beta_1$. Then because $\alpha_0 \geq \beta_0$ and $\alpha_1 \leq \beta_1$ it follows that $\alpha_0 - G(\alpha_0) \geq 0$ and $\alpha_1 - G(\alpha_1) \leq 0$. Thus by the intermediate value theorem there exists $\beta \in I$ such that $\beta - G(\beta) = 0$ i.e. $G(\beta) = \beta$. [3] ■

Now that we all the necessary tools, we will begin the actual proof of the Li&Yorke theorem

Proof [Li&Yorke] Case 1: Assume $d \leq a < b < c$, write $K = [a, b]$ and $L = [b, c]$. Since $F(a) = b$, $F^2(a) = c$, and $F^3(a) = d$, then corollary 4 implies that

$$L = [b, c] = [F(a), F(b)] \subset F[a, b] = F(K), \quad [d, c] = [F(c), F(b)] \subset F[b, c] = F(L)$$

Since $L \subset [d, c]$ and $K \subset [d, c]$ then $L \subset F(L)$ and $K \subset F(L)$.

Case 2: Assume $d \geq a > b > c$, define $K = [b, a]$ and $L = [c, b]$. Again $F(a) = b$, $F^2(a) = c$, and $F^3(a) = d$, by corollary 4 we have

$$L = [c, b] = [F(b), F(a)] \subset F[b, a] = F(K), \quad [c, d] = [F(b), F(c)] \subset F(c, b] = F(L)$$

Since $L \subset [c, d]$ and $K \subset [c, d]$ then $L \subset F(L)$ and $K \subset F(L)$. Thus in both cases we have

$$K \subset F(L), \quad L \subset F(L), \quad L \subset F(K).$$

Let $z \in \mathbb{N}$. For $z \geq 2$ let $\{I_n\}$ be the sequence of compact intervals $I_n = L$ for $n = 0, \ldots, z-2$, and $I_{z-1} = K$, and define I_n to periodic inductively $I_{n+z} = I_n$ for $n = 0, 1, 2, \ldots$. If $z = 1$, let $I_n = L$ for all n. Let Q_n be a sequence of compact intervals as defined in lemma 8. Then we have the properties that $Q_{n+1} \subset Q_n \subset L$ and $F^n(Q_n) = I_n$ for all $n \in \mathbb{N}$. Since $Q_z \subset Q_0$ we see that $Q_z \subset Q_0 = L = I_z = F^z(Q_z)$ then by lemma 10 we see that F^z has a fixed point p_z in Q_z. Now we are going to show by contradiction that this point has period z. Suppose p_z has period $x \in \{0, \ldots, z-1\}$ or in other words $F^x(p_z) = p_z$. By lemma 8 we know that $F^{z-1}(p_z) \in I_{z-1} = K$ and also that $F^m(p_z) \in I_m = L$ for $m < z - 1$. Furthermore, we trivially have $z - 1 - x < z - 1$, this implies that

$$\begin{aligned} F^{z-1-x}(p_z) &= F^{z-1-x}(F^x(p_z)) \\ &= F^{z-1-x+x}(p_z) \\ &= F^{z-1}(p_z) \end{aligned}$$

It follows that $F^{z-1}(p_z) \in L$ but also $F^{z-1}(p_z) \in K$, hence $F^{z-1}(x) \in L \cap K$ thus $F^{z-1}(p_z) = b$. Do note that $F^{z+1}(p_z) = F(p_z) \in L$ and $F^{z+1}(p_z) = F^2(F^{z-1}(p_z)) = F^2(b) = d$. But $d \notin L$, so we have a contradiction. Thus p_z has period z. [3] ■

■

4. Discussion of Ly&Yorke theorem and proof

The Li&Yorke theorem is quite strong. If one finds a periodic point of period three then for every $k \in \mathbb{N}$ there exist a point in the interval which has period k. It also implies that the discrete dynamical system exhibits chaos, as defined in the theorem. Furthermore the restriction to functions is relatively loose, only continuity is required, this makes it so that even simplistic functions can have chaos. We will see later in an example that is can be the case. Moreover, we see that the proof consists of concepts that are relatively fundamental, in the sense that it does not require new abstract concepts. Although the proof of the theorem is relatively simplistic, the result is quite exceptional. This makes the theorem in my opinion even more beautiful.

A question that may arise is: 'Do periodic points of other periods also imply periodic points with a period for a $k \in \mathbb{N}$?'. This question can be answered with a generalization of the Ly&Yorke theorem. is closely related to a deeper and more general result due to Oleksandr Mykolaiovych Sharkovsky. Sharkovsky proved his theorem in 1964 in a paper written in Ukrainian and published in a Soviet journal [9]. Because of the political and linguistic barriers of the Cold War, this result remained largely unknown in the West for more than a decade. As a consequence, Li and Yorke were not aware of Sharkovsky's work when they proved their theorem in 1975. Only later was it recognized that the Li–Yorke result is a consequence of Sharkovsky's theorem. This historical separation explains why both results coexist in the literature and why the phrase "period three implies chaos" became so influential in Western mathematics.

The Sharkovsky Ordering

At the heart of Sharkovsky's theorem lies a ordering of the positive integers, now called the *Sharkovsky ordering*. The ordering is defined as follows:

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \cdots \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec \cdots \prec \cdots \prec 2^k \prec \cdots \prec 2^2 \prec 2 \prec 1.$$

Note that $x \prec y$ then x is higher hierarchy then y. First come all odd integers greater than one in increasing order, here three is of greater hierarchy then 5 etc. Aftet that come all odd intergers greater then one times 2. Then come all odd intergers greater than one times 2^2. This structure goes on and on. Of the lowest hierarchy are the powers of two, here one is of the lowest hierarchy with respect to any other natural number. This ordering is not arbitrary; it reflects the way interval maps can fold and reverse orientation under iteration. Sharkovsky his theorem is based of this ordering of the natural number.

Sharkovsky Theorem

Theorem 11 (Sharkovskiĭ, 1964) *If a continuous map $f : [0,1] \to [0,1]$ has a cycle of minimal period p, then it has a cycle of minimal period q, for any $p \prec q$ [9].*

In particular, since 3 is the first element in the ordering, the existence of a period-3 orbit forces the existence of periodic orbits of *all* other periods. This immediately implies the Li–Yorke theorem. The Sharkovsky theorem is *sharp*. That is, the implications it describes cannot be strengthened. For example, although period 3 implies period 5, the converse is not necessarily true. The existence of a period-5 orbit does not imply the existence of a period-3 orbit. More generally, if $n \prec m$ in the Sharkovskiĭ ordering, then one can construct a continuous map of the interval that has a periodic orbit of period m but no periodic orbit of period n. Thus, the Sharkovskiĭ ordering gives a complete and optimal classification of forced periodic behavior for continuous interval maps.

5. Relation logistic map

One of the most well known examples in one-dimensional discrete dynamical systems is the logistic map. It is defined in the following way

$$f(x, \alpha) = \alpha x(1 - x), \qquad x \in [0, 1],\ \alpha \in (0, \infty).$$

The logistic map was originally introduced as a discrete-time model of population growth. Here x_n represents the normalized population size in generation n and the parameter α represents the intrinsic growth rate [8]. The nonlinear term $x(1 - x)$ models competition for limited resources, leading to saturation effects at high population levels. Mathematically it means that $(1 - x)$ is a correction term that stops the population size from exponentially growing. Despite its simple algebraic form, the logistic map exhibits remarkably rich dynamical behavior as the parameter α varies. For small values of α, the dynamics converge to a stable fixed point, while increasing α leads to a sequence of *bifurcations*, including period-doubling cascades and the eventual onset of chaotic dynamics [10]. This transition from simple to complex behavior makes the logistic map a canonical example in the study of nonlinear dynamics and chaos. The logistic map gained particular prominence following the work of May, who demonstrated that even simple deterministic population models can generate highly complex and unpredictable behavior [8]. Subsequent mathematical analysis revealed that the logistic map contains periodic orbits of arbitrarily large order and displays chaotic dynamics for a wide range of parameter values. A useful way to visualize the dependence of the long-term dynamics of the logistic map on the parameter α is through its *bifurcation diagram*. The bifurcation diagram plots the asymptotic values of the iterates x_n against the parameter α. For each fixed value of α, one considers a large number of iterates after transient behavior has decayed and records the values approached by the orbit. Stable fixed points and periodic orbits appear as discrete points, while chaotic dynamics give rise to dense vertical sets [2],[10]. The bifurcation diagram for the logistic map can be seen in figure 1 on the next page. We are going to give examples of the regimes/bifurcations for certain α. As we go through the different regimes and bifurcations of the logistic map, we encourage the reader to continually refer back to the bifurcation diagram, as it gives a clear visual overview of how the dynamics change with the parameter. For $0 < \alpha < 1$, the origin $x = 0$ is the unique fixed point of the system and is globally attracting. In this parameter regime, all orbits converge to zero, corresponding to extinction in the original population model interpretation [10]. For $1 < \alpha < 3$, the fixed point $x^* = 1 - \frac{1}{\alpha}$ exists and is asymptotically stable, while the origin becomes unstable. This collision at $\alpha = 1$ gives rise to a phenomenon known as a *transcritical bifurcation* [11]. In the range $1 < \alpha < 3$, almost all orbits converge to x^*, which appears in the bifurcation diagram as a single stable branch [2]. At $\alpha = 3$, the logistic map undergoes a *period-doubling bifurcation*. The fixed point x^* loses stability, and a stable periodic orbit of period two is created. As α increases further this period-two orbit undergoes successive period-doubling bifurcations. That produces stable orbits of periods 2^n. This cascade accumulates at $\alpha_\infty \approx 3.56995$, beyond which chaotic dynamics occur for a large set of parameter values [1],[10]. For $\alpha > \alpha_\infty$, the system displays chaotic behavior characterized by sensitive dependence on initial conditions and aperiodic orbits. In the bifurcation diagram, this chaotic regime appears as a broad, densely filled region. Embedded within this chaotic regime are *periodic windows*, where stable periodic orbits temporarily reappear [2]. Of particular interest is the period-three window. At the critical parameter value$\alpha_0 = 1 + 2\sqrt{2}$, the third iterate f^3 undergoes a fold (saddle–node) bifurcation, producing a pair of period-three orbits, one stable and one unstable [1]. This bifurcation is visible in the bifurcation diagram as the sudden appearance of three distinct branches. The appearance of a period-three orbit has implications. By the Li–Yorke theorem, the existence of such an orbit implies the existence of periodic orbits of every period and chaotic dynamics in the sense of Li and Yorke. Thus, the bifurcation diagram provides a visual manifestation of the transition from regular dynamics to chaos in the logistic map [7].

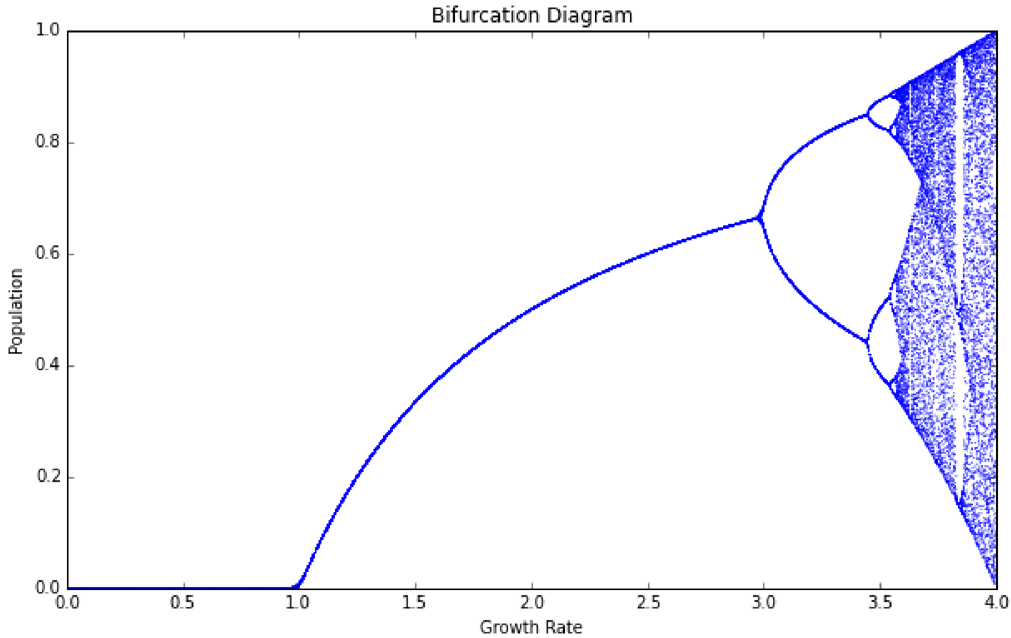

Figure 1: Bifurcation diagram of the logistic map

A crucial property of the logistic map is that for $0 \leq \alpha \leq 4$, the unit interval is forward invariant:

$$f([0,1],\alpha) \subset [0,1].$$

This allows the logistic map to be studied rigorously as a continuous self-map of a compact interval, where tools such as the Intermediate Value Theorem and ordering arguments can be applied [4]. In this section, we analyze the third iterate of the logistic map and prove that at the critical parameter value

$$\alpha_0 = 1 + 2\sqrt{2},$$

Proposition 12 *At $\alpha_0 < 4$ the third iterate of the logistic map exhibits a generic fold bifurcation, generating a stable period-3 cycle and an unstable period-3 cycle of the logistic map as α increases.*

Proof We introduce the function

$$G(x,\alpha) = f^3(x,\alpha)$$

the third iterative of the logistic map. One gets:

$$G(x,\alpha) = \alpha^3 x(1-x)(1-\alpha x+\alpha x^2)(1-\alpha^2 x+\alpha^3 x^2-2\alpha^3 x^3+\alpha^2 x^2+\alpha^3 x^4).$$

A point x belongs to a period-three orbit of f if and only if it is a fixed point of G which is not a fixed point of f or f^2. A fold (saddle–node) bifurcation of a one-dimensional map occurs when two fixed points collide and annihilate or are created. For the map G, this happens when a fixed point becomes non-hyperbolic with multiplier equal to $+1$. Therefore, the fold bifurcation conditions for a period-three cycle translate into the following system:

$$\begin{cases} G(x,\alpha) - x = 0, \\ G_x(x,\alpha) - 1 = 0. \end{cases}$$

The first equation expresses the existence of a fixed point of the third iterate, corresponding to a period-three orbit of the original map. The second equation ensures that the fixed point is non-hyperbolic, which is the defining condition for a fold bifurcation in one-dimensional discrete dynamical systems. One can check that by eliminating x from $G(x, \alpha) - x$ that it reduces to

$$(\alpha^2 - 2\alpha - 7)(\alpha - 1)^2(\alpha^2 + \alpha + 1)^2(\alpha^2 - 5\alpha + 7)^2 = 0,$$

whose real solutions coincide with those of the equation

$$(\alpha^2 - 2\alpha - 7)(\alpha - 1) = 0.$$

The solutions are

$$\alpha_1 = 1 + 2\sqrt{2}, \quad \alpha_2 = 1 - 2\sqrt{2}, \quad \alpha_3 = 1$$

Note that $\alpha_2 < 0$ and is out of our range for α. Furthermore, α_3 is related to the transcritical bifurcation of the fixed point $x = 0$. Hence α_1 is the only critical value. It is straightforward to verify that this equation

$$G(x, \alpha_1) = x.$$

admits two trivial solutions,

$$x = 0 \quad \text{and} \quad x = 1 - \frac{1}{\alpha_1} = \frac{8 - 2\sqrt{2}}{7},$$

which correspond to the fixed points of the original logistic map $f(\cdot, \alpha_1)$. Factoring out these solutions, the fixed-point equation can be written as

$$G(x, \alpha_1) - x = x\,(7x - 8 + 2\sqrt{2})\left(343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2}\right).$$

The cubic factor

$$343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2}$$

has three distinct real roots

$$x_1 < x_2 < x_3.$$

These roots correspond to three distinct fixed points of G, and hence determine a period-three cycle

$$\{x_1, x_2, x_3\}$$

of the original logistic map at $\alpha = \alpha_1$.

To determine the nature of the bifurcation, we evaluate the second derivative and the parameter derivative of G at one of the roots, say x_1. A direct calculation shows that

$$G_{xx}(x_1, \alpha_1) > 0 \quad \text{and} \quad G_\alpha(x_1, \alpha_1) < 0.$$

These inequalities verify the nondegeneracy and transversality conditions of the fold bifurcation theorem. By the fold (saddle–node) bifurcation theorem for one-dimensional maps, these conditions imply that there exists a neighbourhood of (x_1, α_1) in which the fixed-point equation

$$G(x, \alpha) = x$$

has:

- no solutions for $\alpha < \alpha_1$,
- exactly one double solution at $\alpha = \alpha_1$,

- exactly two distinct solutions for $\alpha > \alpha_1$.

Since fixed points of G correspond bijectively to period-three cycles of the original logistic map f, it follows that for $\alpha > \alpha_1$ and sufficiently close to α_1, exactly two period-three cycles are created.

To determine their stability, we recall that a fixed point of a one-dimensional map is stable if

$$|G_x(x, \alpha)| < 1$$

and unstable if

$$|G_x(x, \alpha)| > 1.$$

In a fold bifurcation, the two branches of fixed points emerging for $\alpha > \alpha_1$ lie on opposite sides of the neutral stability condition $G_x = 1$. Consequently, one branch satisfies $|G_x| < 1$ and corresponds to a stable fixed point, while the other satisfies $|G_x| > 1$ and corresponds to an unstable fixed point.

Therefore, for $\alpha > \alpha_1$ sufficiently close to α_1, the logistic map possesses exactly two period-three cycles: one stable and one unstable. [6] ■

6. Summary

We have seen that the Li–Yorke theorem is a very strong result. The existence of a single period-3 orbit in a continuous map guarantees periodic orbits of all other periods. It also implies chaotic behavior in the Li–Yorke sense. The proof uses fundamental and relatively simple concepts, such as fixed points, the intermediate value theorem, and the structure of orbits. This result is closely related to the Sharkovsky theorem. Sharkovsky provides a complete ordering of natural numbers that determines which periods must exist once a given period appears. We then explored the logistic map and its bifurcation diagram. Various bifurcations appear as the parameter increases, including fold and period-doubling bifurcations. By analyzing the third iterate, we showed the existence of a period-3 orbit at a specific parameter value. By Li–Yorke's theorem, this implies the onset of chaos. These results show how even simple one-dimensional maps can exhibit very rich and complex behavior.

For readers interested in exploring further, there are several natural directions. One can study the full Sharkovsky theorem and its proof, which generalizes Li–Yorke and provides a complete classification of forced periods in continuous interval maps. Another direction is to investigate chaos in dynamical systems more broadly, including modern notions such as sensitivity to initial conditions, Lyapunov exponents, and invariant sets. The period-doubling route to chaos in the logistic map is also a rich topic, illustrating how repeated period doublings lead to complex behavior. Finally, one may explore Feigenbaum's constants and the universality of bifurcation cascades, which connect one-dimensional maps to a wide range of systems in physics and mathematics. Each of these areas provides deeper insight into how simple rules can generate intricate and fascinating dynamics.

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