Periodic perturbations of planar Hamiltonian systems

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Introduction to the Melnikov function

We consider the following planar Hamiltonian system

$$\begin{cases} \dot{x} = H_y(x, y) \\ \dot{y} = -H_x(x, y), \end{cases} \tag{1}$$

with $(x, y) \in \mathbb{R}^2$. In this system, $H : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the smooth Hamiltonian function and H_x and H_y denote its partial derivatives with respect to x and y.

Equivalently, in vector form, we can write system (1) as

$$\dot{q} = JDH(q), \tag{2}$$

with $q = (x, y)$, J $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $DH = (H_x, H_y)$.

We assume that the unperturbed Hamiltonian system contains a hyperbolic saddle point $p_0 = (x_0, y_0)$ with a homoclinic orbit $q_0(t) = (x_0(t), y_0(t))$. This homoclinic orbit connects p_0 to itself and satisfies $H(x(t), y(t)) = h_0$, where h_0 is a constant. We indicate the stable manifold of the saddle point with W^s and the unstable manifold with W^u and we let $\Gamma_{p_0} = \{q_0(t) | t \in \mathbb{R}\} \cup \{p_0\} = (W^s(p_0) \cap W^u(p_0)) \cup \{p_0\}$ describe the homoclinic manifold. Let a continuous family of periodic orbits $q^\alpha(t)$ with period $T^\alpha, \alpha \in (-1, 0)$ occupy the interior of Γ_{p_0}. We assume that $\lim_{\alpha \to 0} q^\alpha(t) = q^0(t)$ and $\lim_{\alpha \to 0} T^\alpha = \infty$.

We consider the following class of perturbed Hamiltonian systems

$$\begin{cases} \dot{x} = H_y(x, y) + \epsilon f_1(x, y, t, \epsilon) \\ \dot{y} = -H_x(x, y) + \epsilon f_2(x, y, t, \epsilon), \end{cases} \tag{3}$$

where $f_1(x, y, t, \epsilon)$ and $f_2(x, y, t, \epsilon)$ are periodic functions in t with periodicity $T = \frac{2\pi}{\omega}$, and ϵ is a small parameter. Writing system (3) in vector notation, yields the system

$$\dot{q} = JDH(q) + \epsilon f(q, t, \epsilon), \tag{4}$$

with $f = (f_1, f_2)$. It is assumed that (4) is sufficiently differentiable on the area of our concern. In this context, $\mathbf{C}^r$ with $r \geq 2$ will suffice. Setting $\epsilon = 0$ reduces system (4) to the unperturbed Hamiltonian vector field (2).

Melnikov's method is a technique used to detect chaos in dynamical systems. The Melnikov function is important in the context of perturbations of Hamiltonian systems. In particular, it is useful in the study of the onset of chaotic dynamics, by way of assessing whether the perturbations cause qualitative changes in the behaviour of a system. The Melnikov function quantifies the distance between the stable and unstable manifolds of a perturbed dynamical system such that both the magnitude and the direction of the separation are captured in the equation.

Parametrization of the homoclinic manifold in the unperturbed system

Before we introduce the Melnikov function, we develop a parametrization of the homoclinic orbit of the unperturbed Hamiltonian system. In order to do so, we rewrite system (4) to the three-dimensional system

$$\begin{cases} \dot{q} = JDH(q) + \epsilon f(q, \phi, \epsilon) \\ \dot{\phi} = \omega, \end{cases} \tag{5}$$

with $(q, \phi) \in \mathbb{R}^2 \times S^1$. For $\epsilon = 0$, system (5) reduces to the unperturbed system

$$\begin{cases} \dot{q} = JDH(q) \\ \dot{\phi} = \omega. \end{cases} \tag{6}$$

The hyperbolic saddle point p_0 of system (6) of the q component gives rise to a periodic orbit $\gamma(t) = (p_0, \phi(t) = \omega t + \phi_0)$ in the unperturbed system. In the phase space $\mathbb{R}^2 \times S^1$, the unstable and stable manifolds of $\gamma(t)$ are given by $W^u(\gamma(t))$ and $W^s(\gamma(t))$, respectively. Since we assume there is a homoclinic orbit to the saddle in the unperturbed system, $W^u(\gamma(t))$ and $W^s(\gamma(t))$ unite and form a single homoclinic manifold Γ_γ, see Figure 1

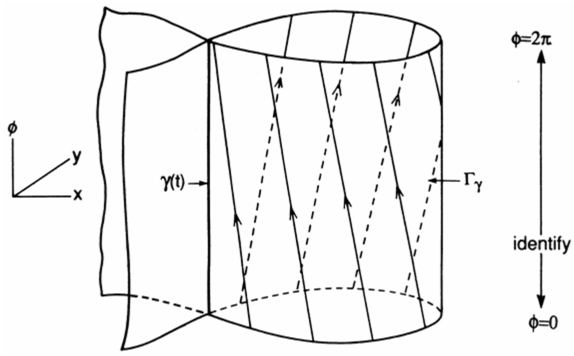

Figure 1: The homoclinic manifold Γ_γ with the periodic orbit $\gamma(t)$ (Fig 28.1.2 from [1]).

The points $q_0(t) = (x_0(t), y_0(t))$ for $t \in \mathbb{R}$ represent the position of the system at time t and precisely follow the homoclinic orbit in the unperturbed system, connecting the saddle point to itself. Each point $p \in \Gamma_\gamma$ can be expressed as

$$p = (q_0(-t_0), \phi_0), \tag{7}$$

with $t_0 \in \mathbb{R}$ and $\phi_0 \in (0, 2\pi]$. In this notation, we interpret t_0 as the time of flight from the point $q_0(-t_0)$ to $q_0(0)$. Thus we can write

$$\Gamma_\gamma = \{(q_0(-t_0), \phi_0) \in \mathbb{R}^2 \times S^1 | t_0 \in \mathbb{R}, \phi_0 \in (0, 2\pi]\}. \tag{8}$$

Splitting of the manifolds in the perturbed system

Next, we want to define the separation of the unstable and the stable manifold in the pertubed system 5. Our aim is to investigate how the homoclinic manifold Γ_γ is affected by the perturbation. First, let us focus on two essential perturbation results. That is, for small $\epsilon > 0$, the periodic orbit $\gamma(t)$ of unperturbed system 6 persists as a periodic orbit in perturbed system 5 as $\gamma_\epsilon(t) = \gamma(t) + \mathcal{O}(\epsilon)$ with the same stability type as $\gamma(t)$. Furthermore, the local stable and unstable manifolds $W^s_{loc}(\gamma_\epsilon(t))$ and $W^u_{loc}(\gamma_\epsilon(t))$ of the perturbed periodic orbits are $\mathbf{C}^r$ ϵ-close to those of the unperturbed periodic orbit $W^s_{loc}(\gamma(t))$ and $W^u_{loc}(\gamma(t))$.

This means that, for ϵ_0 sufficiently small, the periodic orbit $\gamma_\epsilon(t)$ of the perturbed vector field is contained in the small neighborhood $\mathcal{N}(\epsilon_0)$ containing $\gamma(t)$, with a distance $\mathcal{O}(\epsilon_0)$ from $\gamma(t)$ to the boundary of $\mathcal{N}(\epsilon_0)$. For $0 < \epsilon < \epsilon_0$ we can write

$$W^{s,u}_{loc}(\gamma_\epsilon(t)) = W^{s,u}_{loc}(\gamma(t)) \cap \mathcal{N}(\epsilon_0). \tag{9}$$

To verify these results, we introduce the cross-section of the phase space

$$\Sigma^{\phi_0} = \{(q, \phi) \in \mathbb{R}^2 \times S^1 | \phi = \phi_0\} \tag{10}$$

which is parallel to the q-plane. The dynamics are captured by the Poincaré map

$$\begin{aligned} P_\epsilon^{\phi_0} :&\Sigma^{\phi_0} \to \Sigma^{\phi_0} \\ &q_\epsilon(0) \mapsto q_\epsilon(2\pi/\omega). \end{aligned} \tag{11}$$

Since the periodic orbit $\gamma(t)$ intersects Σ^{ϕ_0} at the point $= p_0$, it follows that $\gamma(t)$ corresponds to a fixed point of the unperturbed Poincaré map, so $P_0^{\phi_0}(p_0) = p_0$. We know that the unperturbed system contains a hyperbolic periodic orbit $\gamma(t)$, so p_0 is a hyperbolic fixed point of $P_0^{\phi_0}$. We define $F_\epsilon(x) = P_\epsilon^{\phi_0}(x) - x$ and for the unperturbed system, $x = p_0$ satisfies $F_0(p_0) = 0$. We want to show that the fixed point p_0 of the unperturbed Poincaré map $P_0^{\phi_0}$ persists as a fixed point p_ϵ^* of the perturbed Poincaré map $P_\epsilon^{\phi_0}$. By the implicit function theorem, a solution $x = p_\epsilon^*$ to $F_\epsilon(x) = 0$ exist for sufficiently small ϵ, provided that $F_\epsilon(x)$ is sufficiently smooth ($\mathbf{C}^r, r \geq 1$) in x and ϵ and that the Jacobian $D_x F_0(p_0)$ is invertible. As hyperbolicty implies that the Poincaré map near $\gamma(t)$ has no eigenvalues on the unit circle and since $D_x F_0(p_0) = DP_0^{\phi_0}(p_0) - I$, we find that $D_x F_0(p_0)$ is invertible. Thus, there exists a unique fixed point $\epsilon \mapsto p_\epsilon^*$ that satisfies $P_\epsilon^{\phi_0}(p_\epsilon^*) = p_\epsilon^*$. As a fixed point p_ϵ^* of the Poincaré map corresponds directly to a periodic orbit $\gamma_\epsilon(t)$ of the perturbed system and considering p_ϵ^* is $\mathcal{O}(\epsilon)$ close to p_0, this implies the existence of $\gamma_\epsilon(t) = \gamma(t) + \mathcal{O}(\epsilon)$. Since P_ϵ is $\mathbf{C}^r$-close to P_0, so are the eigenvalues and the stability type remains the same. By the stable and unstable manifold theorem for maps [2] $W^s_{loc}(\gamma_\epsilon(t))$ and $W^u_{loc}(\gamma_\epsilon(t))$ are $\mathcal{O}(\epsilon)$-close to $W^s_{loc}(\gamma(t))$ and $W^u_{loc}(\gamma(t))$, respectively.

To construct the splitting of $W^u(\gamma_\epsilon(t))$ and $W^s(\gamma_\epsilon(t))$, we focus on $p \in \Gamma_\gamma$. For the purpose of measuring the distance between the pertubed stable and unstable manifolds at a point p in the homoclinc manifold, we want to introduce a vector in the direction normal to Γ_γ. Thus, we introduce the vector

$$\pi_p = (DH(q_0(-t_0)), 0). \tag{12}$$

It is important to note that at every point $p \in \Gamma_\gamma$, the stable and unstable manifolds $W^s(\gamma(t))$ and $W^u(\gamma(t))$ intersect the vector π_p transversely at point p. For any point $p_w \in W^s \cap W^u$, the sets W^s and W^u are said to be transversal at p_w if $T_{p_w}W^s + T_{p_w}W^u = \mathbb{R}^3$. Here, $T_{p_w}W^s$ and $T_{p_w}W^u$ denote the tangent spaces of the stable and unstable manifolds, respectively, at the point p_w [1]. Since transversality persists and as we assume that $W^u(\gamma(t))$ and $W^s(\gamma(t))$ are sufficiently smooth ($\mathbf{C}^r$ with $r \geq 2$), for ϵ small enough, we state that the separated manifolds $W^u(\gamma_\epsilon(t))$ and $W^s(\gamma_\epsilon(t))$ cross π_p transversely in the points p_ϵ^u and p_ϵ^s correspondingly. Consequently, the distance between $W^u(\gamma_\epsilon(t))$ and $W^s(\gamma_\epsilon(t))$ at the point p, denoted by $d(p, \epsilon)$, can be derived from the distance between p_ϵ^u and p_ϵ^s as follows

$$d(p, \epsilon) = |p_\epsilon^u - p_\epsilon^s|. \tag{13}$$

We can rewrite this expression (13) to

$$d(p, \epsilon) = \frac{(DH(q_0(-t_0), 0) \cdot (p_\epsilon^u - p_\epsilon^s)}{\|(DH(q_0(-t_0))\|} = \frac{\pi_p \cdot (p_\epsilon^u - p_\epsilon^s)}{\|(DH(q_0(-t_0))\|}, \tag{14}$$

where we compute the vector scalar product "$\cdot$" and the norm "$\|.\|$" of the vector $DH(q_0(-t_0))$. Since the points p_ϵ^u and p_ϵ^s both lie on the vector $(DH(q_0(-t_0)), 0)$, it is clear that (13) is equivalent to (14). This analogous expression of the distance between $W^u(\gamma_\epsilon(t))$ and $W^s(\gamma_\epsilon(t))$ will prove to be useful henceforth.

By definition, p_ϵ^u and p_ϵ^s have the same ϕ_0 coordinate and therefore, we can write $p_\epsilon^s = (q_\epsilon^s, \phi_0)$ and $p_\epsilon^u = (q_\epsilon^u, \phi_0)$. Thus, (14) is equivalent to

$$d(t_0, \phi_0, \epsilon) = \frac{(DH(q_0(-t_0)) \cdot (q_\epsilon^u - q_\epsilon^s)}{\|(DH(q_0(-t_0))\|}, \tag{15}$$

since every $p \in \Gamma_\gamma$ can be expressed in parameters $t_0 \in \mathbb{R}, \phi_0 \in (0, 2\pi]$ as in 7.

Before we can define the Melnikov function using expression (15), we concentrate on a technical issue regarding the choice of the points p_ϵ^s and p_ϵ^u, where $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ cross π_p transversely. These manifolds may cross the vector π_p more than once. In this case, it is not yet clearly defined as to which points will be chosen for p_ϵ^u and p_ϵ^s so as to define the distance function 15. Thus, we state a definition for the points closest to $\gamma_\epsilon(t)$ in terms of the time of flight along the stable or unstable manifold.

Definition. *Let us define points $p^s_{\epsilon,i} \in W^s(\gamma_\epsilon(t)) \cap \pi_p$ and $p^u_{\epsilon,i} \in W^u(\gamma_\epsilon(t)) \cap \pi_p$, for i in some index set $\mathcal{I}$. Orbits of the perturbed vector field are denoted with $(q^s_{\epsilon,i}(t), \phi(t)) \in W^s(\gamma_\epsilon(t))$ and $(q^u_{\epsilon,i}(t), \phi(t)) \in W^u(\gamma_\epsilon(t))$, with the initial conditions $(q^s_{\epsilon,i}(0), \phi(0)) = p^s_{\epsilon,i}$ and $(q^u_{\epsilon,i}(0), \phi(0)) = p^u_{\epsilon,i}$, respectively.*

Then, a point $p^s_{\epsilon,\tilde{i}}$ is considered to be the closest point in $W^s(\gamma_\epsilon(t))$ to $\gamma_\epsilon(t)$ in terms of positive time of flight, if for all $t > 0$, the trajectory $(q^s_{\epsilon,i}(t), \phi(t))$ does not intersect π_p. Similarly, a point $p^u_{\epsilon,\tilde{i}}$ is the closest point in $W^u(\gamma_\epsilon(t))$ to $\gamma_\epsilon(t)$ in terms of negative time of flight, if for all $t < 0$, the intersection between the trajectory $(q^u_{\epsilon,i}(t), \phi(t))$ and π_p is empty [1].

For a visual interpretation, see Figure 2.

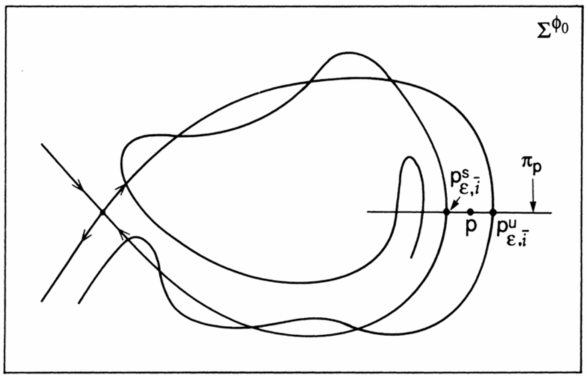

Figure 2: The points $p^s_{\epsilon,\tilde{i}}$ and $p^u_{\epsilon,\tilde{i}}$ closest to $\gamma_\epsilon(t)$ in terms of negative time of flight along $W^s(\gamma_\epsilon(t))$ and posiftive time of flight along $W^u(\gamma_\epsilon(t))$, respectively, are shown (Fig 28.1.12 from [1]).

In our definition of the distance (13) between the stable and the unstable manifold of the perturbed periodic orbit $\gamma_\epsilon(t)$ at point p 13, we choose the unique points $p^s_\epsilon = p^s_{\epsilon,\tilde{i}}$, closest to $\gamma_\epsilon(t)$ in terms of positive time of flight along $W^s(\gamma_\epsilon(t))$, and $p^u_\epsilon = p^u_{\epsilon,\tilde{i}}$ closest in terms of negative time of flight along $W^u(\gamma_\epsilon(t))$.

Deriving the Melnikov function

Subsequently, we want to derive the Melnikov function, starting with a Taylor expansion of (15) about $\epsilon = 0$. This gives

$$d(t_0, \phi_0, \epsilon) = d(t_0, \phi_0, 0) + \epsilon \frac{\partial d}{\partial \epsilon}(t_0, \phi_0, 0) + \mathcal{O}(\epsilon^2). \tag{16}$$

In this expression, $d(t_0, \phi_0, 0)$ reduces to 0 and

$$\frac{\partial d}{\partial \epsilon}(t_0, \phi_0, 0) = \frac{(DH(q_0(-t_0)) \cdot \left(\frac{\partial q^u_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0} - \frac{\partial q^s_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0} \right)}{\|(DH(q_0(-t_0))\|} = \frac{M(t_0, \phi_0)}{\|(DH(q_0(-t_0))\|}, \tag{17}$$

where $M(t_0, \phi_0)$ represents the Melnikov function.

We note that $DH(q_0(-t_0)) = (H_x(q_0(-t_0)), H_y(q_0(-t_0))$ is never equal to 0 on $q_0(-t_0)$. The point $q_0(-t_0)$ lies on the homoclinic orbit in the unperturbed system. At q_0, the system is moving along the orbit towards the saddle point. This means that motion occurs, thus either $H_x \neq 0$ or $H_y \neq 0$, as these derivatives determine the velocity of the system. It follows that, for a finite t_0, if the Melnikov function $M(t_0, \phi_0)$ approaches zero, then $\frac{\partial d}{\partial \epsilon}(t_0, \phi_0, 0) = 0$. Thus, aside from the nonzero scaling factor $\|(DH(q_0(-t_0))\|$, the Melnikov function is constructed as the first nonzero term in the Taylor expansion of the distance between $W^u(\gamma_\epsilon(t))$ and $W^s(\gamma_\epsilon(t))$ at the point p.

In order to express the Melnikov function such that the definition is not dependent on a solution of the perturbed vector field, we want to redefine the Melnikov function $M(t_0, \phi_0)$ in a way that it depends on time by using the flow that is produced by the perturbed vector field as well as the unperturbed vector field. Subsequently, we derive an ordinary differential equation which this time-dependent function must satisfy. The solution of this differential equation at the proper time yields the final Melnikov function. Orbits in the stable manifold $W^s(\gamma_\epsilon(t))$ are referred to as $q^s_\epsilon(t)$. Similarly, orbits in the unstable manifold $W^u(\gamma_\epsilon(t))$ are denoted by $q^u_\epsilon(t)$. We define the time-dependent Melnikov function as

$$M(t; t_0, \phi_0) = DH(q_0(t-t_0)) \cdot \Big(\frac{\partial q^u_\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0} - \frac{\partial q^s_\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0} \Big), \tag{18}$$

where $M(0; t_0, \phi_0) = M(t_0, \phi_0)$. Equation 18 contains the partial derivatives of $q^s_\epsilon(t)$ and $q^u_\epsilon(t)$ with respect to ϵ, evaluated at $\epsilon = 0$, which evolve in time under the dynamics of the perturbed system. Observe the expression $DHq_0(t-t_0))$ is also contained in $M(t; t_0, \phi_0)$ and advances in time under the dynamics of the unperturbed vector field. To shorten the notation, we define

$$\begin{aligned} \frac{\partial q^s_\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0} &= q^s_d(t) \\ \frac{\partial q^u_\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0} &= q^u_d(t) \end{aligned} \tag{19}$$

and

$$\Delta^{s,u}(t) = DH(q_0(t-t_0)) \cdot q^{s,u}_d(t). \tag{20}$$

Now, we can write equation (18) as

$$M(t; t_0, \phi_0) = DH(q_0(t-t_0)) \cdot (q^u_d(t) - q^s_d(t)) = \Delta^u(t) - \Delta^s(t). \tag{21}$$

Next, to define our final form of the Melnikov function, we differentiate expression $\Delta^{s,u}(t)$ with respect to t. Using the chain rule, we find

$$\frac{d\Delta^{s,u}(t)}{dt} = \frac{d(DH(q_0(t-t_0)))}{dt} \cdot q^{s,u}_d(t) + DH(q_0(t-t_0)) \cdot \frac{dq^{s,u}_d(t)}{dt}. \tag{22}$$

Recall that $q^{s,u}_\epsilon(t)$ solves

$$\frac{dq^{s,u}_\epsilon(t)}{dt} = JDH(q^{s,u}_\epsilon(t)) + \epsilon f(q^{s,u}_\epsilon(t), \phi(t), \epsilon) \tag{23}$$

in system (5), where $\phi(t) = \omega t + \phi_0$. Since we assume that $q^{s,u}_\epsilon(t)$ is sufficiently smooth in both ϵ and in t, we are allowed to interchange the order of differentiation between ϵ and t. Differentiation of equation (23) with respect to ϵ, by applying the chain rule, gives

$$\frac{dq^{s,u}_d(t)}{dt} = JD^2H(q_0(t-t_0))q^{s,u}_d(t) + f(q_0(t-t_0), \phi(t), 0). \tag{24}$$

Note that q^u_d solves (24) for $t \in (-\infty, 0]$ and q^s_d for $t \in [0, \infty)$. Now that we have found an expression for $\frac{d}{dt}q^{s,u}_d(t)$, we substitute (24) into equation (22) to obtain

$$\frac{d\Delta^{s,u}(t)}{dt} = \frac{dDH(q_0(t-t_0))}{dt} \cdot q^{s,u}_d(t) + DH(q_0(t-t_0)) \cdot JD^2H(q_0(t-t_0))q^{s,u}_d(t) + DH(q_0(t-t_0)) \cdot f(q_0(t-t_0), \phi(t), 0). \tag{25}$$

Let us note that

$$\begin{aligned} \frac{d(DH(q_0(t-t_0)))}{dt} &= D^2H(q_0(t-t_0)\frac{dq_0(t-t_0)}{dt} \\ &= (D^2H(q_0(t-t_0))(JDH(q_0(t-t_0))), \end{aligned}$$

which follows from substitution of system 5 with $\epsilon = 0$ for q_0. We can decompose $q^{s,u}_d(t)$ into $(x^{s,u}_d(t), y^{s,u}_d(t))$ such that computing the first term in (25) gives

$$
\begin{aligned}
(D^2H)(JDH)\cdot q_d^{s,u} &= \begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x\partial y} \\ \frac{\partial^2 H}{\partial y\partial x} & \frac{\partial^2 H}{\partial y^2} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} x_d^{s,u} \\ y_d^{s,u} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial^2 H}{\partial x^2}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial x\partial y}\frac{\partial H}{\partial x} \\ \frac{\partial^2 H}{\partial y\partial x}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial y^2}\frac{\partial H}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} x_d^{s,u} \\ y_d^{s,u} \end{pmatrix} \\
&= x_d^{s,u}\left[\frac{\partial^2 H}{\partial x^2}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial x\partial y}\frac{\partial H}{\partial x}\right] + y_d^{s,u}\left[\frac{\partial^2 H}{\partial y\partial x}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial y^2}\frac{\partial H}{\partial x}\right].
\end{aligned} \tag{26}
$$

Also, we find that the second term in (25) translates to

$$
\begin{aligned}
DH\cdot(JD^2H)q_d^{s,u} &= \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^2 H}{\partial y\partial x} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x\partial y} \end{pmatrix} \begin{pmatrix} x_d^{s,u} \\ y_d^{s,u} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^2 H}{\partial y\partial x}x_d^{s,u} + \frac{\partial^2 H}{\partial y^2}y_d^{s,u} \\ -\frac{\partial^2 H}{\partial x^2}x_d^{s,u} - \frac{\partial^2 H}{\partial x\partial y}y_d^{s,u} \end{pmatrix} \\
&= x_d^{s,u}\left[\frac{\partial^2 H}{\partial y\partial x}\frac{\partial H}{\partial x} - \frac{\partial^2 H}{\partial x^2}\frac{\partial H}{\partial y}\right] + y_d^{s,u}\left[\frac{\partial^2 H}{\partial y^2}\frac{\partial H}{\partial x} - \frac{\partial^2 H}{\partial x\partial y}\frac{\partial H}{\partial y}\right].
\end{aligned} \tag{27}
$$

Addition of (26) and (27) confirms that the first two terms in expression (25) cancel each other, so

$$
\frac{dDH(q_0(t-t_0))}{dt}\cdot q_d^{s,u}(t) + DH(q_0(t-t_0))\cdot JD^2H(q_0(t-t_0))q_d^{s,u}(t) = 0. \tag{28}
$$

Accordingly, expression (25) reduces to

$$
\frac{d\Delta^{s,u}(t)}{dt} = DH(q_0(t-t_0))\cdot f(q_0(t-t_0),\phi(t),0). \tag{29}
$$

Recall that from equation (21), we have

$$
M(t_0,\phi_0) = M(0;t_0,\phi_0) = \Delta^u(0) - \Delta^s(0). \tag{30}
$$

Subsequently, we choose to integrate $\Delta^u(t)$ from $-\tau$ to 0 and $\Delta^s(t)$ from 0 to τ for some $\tau > 0$ in order to substitute these expressions into (30).

As we substitute $\phi(t) = \omega t + \phi_0$ into the integrands, we obtain

$$
\Delta^u(0) - \Delta^u(-\tau) = \int_{-\tau}^{0} DH(q_0(t-t_0))\cdot f(q_0(t-t_0),\omega t+\phi_0,0)dt \tag{31}
$$

and

$$
\Delta^s(\tau) - \Delta^s(0) = \int_{0}^{\tau} DH(q_0(t-t_0))\cdot f(q_0(t-t_0),\omega t+\phi_0,0)dt \tag{32}
$$

Adding expressions (31) and (32) yields

$$
M(t_0,\phi_0) = \int_{-\tau}^{\tau} DH(q_0(t-t_0))\cdot f(q_0(t-t_0),\omega t+\phi_0,0)dt - \Delta^s(\tau) + \Delta^u(-\tau). \tag{33}
$$

Now, we have almost reached our desired expression to define the Melnikov function. We carry on by investigating the limit of 33 as $\tau \to \infty$. Recall that $\Delta^{s,u}(\tau)$ is defined as $DH(q_0(\tau - t_0))\cdot q_d^{s,u}(\tau)$. We see that as τ goes to $\pm\infty$, $DH(q_0(t-t_0))$ will set off to zero. For, as $\tau \to \pm\infty$, the trajectory $q_0(\tau - t_0)$ asymptotically approaches the hyperbolic fixed point p_0. As the gradient of the Hamiltian vanishes at the fixed point p_0, it follows that $DH(q_0(t-t_0) \to DH(p_0) = 0$. Furthermore, as τ goes to ∞ ($-\infty$, respectively), we note that $q_d^s(\tau)$ ($q_d^u(\tau)$, respectively) is bounded. This ensures that as $\tau \to \infty$, $\Delta^{s,u}(\tau) \to 0$.

It follows that, condsidering the limit of 33 as $\tau \to \infty$, the expression reduces to the improper integral

$$
M(t_0,\phi_0) = \int_{-\infty}^{\infty} DH(q_0(t-t_0))\cdot f(q_0(t-t_0),\omega t+\phi_0,0)dt. \tag{34}
$$

This integral is absolutely convergent, as $f(q_0(t-t_0),\omega t+\phi_0,0)$ is bounded for all $t \in \mathbb{R}$ and again, $DH(q_0(t-t_0) \to 0$ as $t \to \pm\infty$.

Properties of the Melnikov function

To complete the introduction to the Melnikov function, a few interesting properties will be considered. Firstly, we note that under the transformation $t \longrightarrow t + t_0$ the Melnikov function (34) converts to

$$M(t_0, \phi_0) = \int_{-\infty}^{\infty} DH(q_0(t)) \cdot f(q_0(t), \omega(t + t_0) + \phi_0, 0)dt. \tag{35}$$

Now, we recall that the function $f(q, t, \epsilon)$ is periodic in t. Thus, it follows that $M(t_0, \phi_0)$ is periodic in t_0 with period $2\pi/\omega$ and periodic in ϕ_0 with period 2π. From expression 35, we see that varying t_0 and ϕ_0 provide the same results, as t_0 is multiplied with ω. As a result, we obtain the following property:

$$\frac{\partial M(t_0, \phi_0)}{\partial t_0} = \omega \frac{\partial M(t_0, \phi_0)}{\partial \phi_0}. \tag{36}$$

Correspondingly, we observe that $\frac{\partial M(t_0,\phi_0)}{\partial t_0} = 0$ if and only if $\frac{\partial M(t_0,\phi_0)}{\partial \phi_0} = 0$.

Secondly, as mentioned before, the Melnikov function is a signed measure of the distance between stable and unstable manifolds as it provides not only the magnitude of the separation, but also indicates the direction of the disengagement. As seen before in (15) , (16) and (17), the distance between $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ at the point $p = (q_0(-t_0), \phi_0)$ is given by

$$\begin{aligned} d(t_0, \phi_0, \epsilon) &= \frac{(DH(q_0(-t_0)) \cdot (q_\epsilon^u - q_\epsilon^s)}{\|(DH(q_0(-t_0))\|} \\ &= \epsilon \frac{M(t_0, \phi_0)}{\|(DH(q_0(-t_0))\|} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{37}$$

Thus, we note that for ϵ sufficiently small, $M(t_0, \phi_0) > 0$ implies $d(t_0, \phi_0, \epsilon) > 0$. As $d(t_0, \phi_0, \epsilon) > 0$ connotes that $q_\epsilon^u > q_\epsilon^s$, these inequalities tell us that the unstable manifold curves wraps around the stable manifold on the outside, see Figure 3. Equivalently, $M(t_0, \phi_0) < 0$ implies $d(t_0, \phi_0, \epsilon) < 0$ and $q_\epsilon^u < q_\epsilon^s$. Thus, in this case, it is the stable manifold that bends along the outside of the unstable manifold.

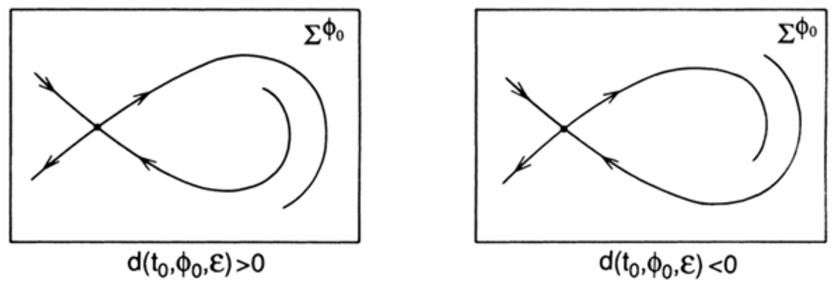

Figure 3: Relative orientations of $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ near p. On the left, $M(t_0, \phi_0) > 0$ and on the right $M(t_0, \phi_0) < 0$ (Fig 28.1.10 from [1]).

Thirdly, if we presume that the perturbation in our system is autonomous, so $\epsilon f(q, \epsilon)$ does not depend explicitly on t, then the Melnikov function is given by

$$M = \int_{-\infty}^{\infty} DH(q_0(t)) \cdot f(q_0(t), 0)dt. \tag{38}$$

We find that M is a number instead of a function of the initial conditions t_0 and ϕ_0. In autonomous two-dimensional vector fields, the stable and unstable manifolds of a hyperbolic saddle point either coincide to form a homoclinic orbit when $M = 0$, or do not intersect at all.

Lastly, we consider the case where the vector field is Hamiltonian, so the perturbation f is derived from a time-dependent Hamiltonian function $\tilde{H}(x, y)$. The perturbed vector field is given by

$$\begin{cases} \dot{x} = H_y(x, y) + \epsilon \tilde{H}_y(x, y, t, \epsilon) \\ \dot{y} = -H_x(x, y) - \epsilon \tilde{H}_x(x, y, t, \epsilon), \end{cases} \tag{39}$$

with the $\mathbf{C}^{r+1}$ function $H_\epsilon(x,y,t) = H(x,y) + \epsilon\tilde{H}(x,y,t,\epsilon)$ periodic in t with period $T = \frac{2\pi}{\omega}$. The corresponding Melnikov function is

$$\begin{aligned} M(t_0,\phi_0) &= \int_{-\infty}^{\infty} DH(q_0(t)) \cdot f(q_0(t), \omega(t+t_0)+\phi_0, 0)dt \\ &= \int_{-\infty}^{\infty} \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \tilde{H}}{\partial y} \\ -\frac{\partial \tilde{H}}{\partial x} \end{pmatrix} (q_0(t), \omega(t+t_0)+\phi_0, 0)dt \\ &= \int_{-\infty}^{\infty} \{H, \tilde{H}\}(q_0(t), \omega(t+t_0)+\phi_0, 0)d, \end{aligned} \tag{40}$$

where $\{H,\tilde{H}\} = H_x\tilde{H}_y - H_y\tilde{H}_x$ denotes the Poisson bracket of H with $\tilde{H}$.

Existence of a transverse homoclinic orbit to the saddle periodic orbit

This section is dedicated to the proof of the following Theorem.

Theorem. *Suppose there is a point $(t_0,\phi_0) = (\bar{t}_0, \bar{\phi}_0)$ where the Melnikov function has a simple zero, so $M(\bar{t}_0,\bar{\phi}_0) = 0$ and $\frac{\partial M(\bar{t}_0,\bar{\phi}_0)}{\partial t_0} \neq 0$. Then for ϵ sufficiently small, the stable and unstable manifolds of $\gamma_\epsilon(t)$, $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$, intersect transversely.*

With this Theorem, we aim to prove the existence of a transverse homoclinic orbit to this cycle. As the Melnikov function is used to measure the distance between the stable and unstable manifolds in a perturbed system, a non-degenerate root of this function indicates a transverse intersection of these manifolds. The two manifolds meet at a point and cross each other with a non-zero angle. For a simple zero of the Melnikov function, we require that at $(\bar{t}_0, \bar{\phi}_0)$ the function $M(t_0,\phi_0)$ vanishes and the Jacobian of $M(t_0,\phi_0)$, given by $\left(\frac{\partial M}{\partial t_0}, \frac{\partial M}{\partial \phi_0}\right)$ is not equal to $(0,0)$. In consequence of the property that $\frac{\partial M(t_0,\phi_0)}{\partial t_0} = 0$ if and only if $\frac{\partial M(t_0,\phi_0)}{\partial \phi_0} = 0$, if either of the partial derivatives is nonzero, then so is the other. A choice was made to state the theorem in terms of $\frac{M(t_0,\phi_0)}{\partial t_0} \neq 0$.

Let us recall from (16) and (17) that we defined the distance between $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ at point $p \in \Gamma_\gamma$ through the distance function

$$\begin{aligned} d(t_0,\phi_0,\epsilon) &= \epsilon \frac{\partial d}{\partial \epsilon}(t_0,\phi_0,0) + \mathcal{O}(\epsilon^2) \\ &= \epsilon \frac{M(t_0,\phi_0)}{\|(DH(q_0(-t_0))\|} + \mathcal{O}(\epsilon^2), \end{aligned}$$

where each point p is parametrized in accordance with $p = (q_0(-t_0), \phi_0)$ for $t_0 \in \mathbb{R}$ and $\phi_0 \in (0, 2\pi]$.

Now, we define

$$\tilde{d}(t_0,\phi_0,\epsilon) = \frac{M(t_0,\phi_0)}{\|(DH(q_0(-t_0))\|} + \mathcal{O}(\epsilon), \tag{41}$$

such that $d(t_0,\phi_0,\epsilon) = \epsilon\tilde{d}(t_0,\phi_0,\epsilon)$. It is evident that $\tilde{d}(t_0,\phi_0,\epsilon) = 0$ implies that $d(t_0,\phi_0,\epsilon) = 0$. Thus we continue our argument with the reduced form $\tilde{d}(t_0,\phi_0,\epsilon) = 0$. Suppose there exists a point $(t_0,\phi_0) = (\bar{t}_0,\bar{\phi}_0)$ where a simple zero of the Melnikov function occurs, hence $M(\bar{t}_0,\bar{\phi}_0) = 0$ and $\left.\frac{\partial M}{\partial t_0}\right|_{(\bar{t}_0,\bar{\phi}_0)} \neq 0$. Then at the point$(t_0,\phi_0,\epsilon) = (\bar{t}_0,\bar{\phi}_0,0)$, expression (41) reduces to

$$\tilde{d}(\bar{t}_0,\bar{\phi}_0,0) = \frac{M(\bar{t}_0,\bar{\phi}_0)}{\|(DH(q_0(-\bar{t}_0))\|} = 0,$$

and

$$\frac{\partial \tilde{d}(\bar{t}_0,\bar{\phi}_0,0)}{\partial t_0} = \frac{1}{\|(DH(q_0(-\bar{t}_0))\|} \frac{\partial M(\bar{t}_0,\bar{\phi}_0)}{\partial t_0} \neq 0.$$

This prerequisite ensures that $\tilde{d}(\bar{t}_0, \bar{\phi}_0, \epsilon = 0)$ can be solved locally, so for small deviations from ϕ_0 and for small values of ϵ, for t_0 as a function of ϕ_0 and ϵ. Henceforth, by the implicit function theorem, it follows that for $|\phi - \phi_0| < \epsilon$, ϵ sufficiently small, there exists a function $t_0 = t_0(\phi_0, \epsilon)$ such that

$$\tilde{d}(t_0(\phi_0, \epsilon), \phi_0, \epsilon) = 0.$$

This expresssion implies that $d(t_0(\phi_0, \epsilon), \phi_0, \epsilon) = 0$, which proves that $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ intersect $\mathcal{O}(\epsilon)$ close.

At this time, all that remains to show is the transversality of the intersection of the stable and unstable manifolds. Recall that for any point $p \in W^s \cap W^u$, this intersection between the sets is transversal if $T_pW^s + T_pW^u = \mathbb{R}^3$, where $T_{p_w}W^s$ and $T_{p_w}W^u$ denote the tangent spaces of the stable and unstable manifolds at p, respectively [1].

Note that, by definition, $p = (q^s_\epsilon, \phi_0) = (q^u_\epsilon, \phi_0)$, the point where the stable and the unstable manifolds intersect. In this regard, $q^{s,u}_\epsilon(t)$ are trajectories in $W^{s,u}(\gamma_\epsilon(t))$, where $q^{s,u}_\epsilon$ are points that are compliant with $q^{s,u}_\epsilon = q^{s,u}_\epsilon(0)$. The tangent space T_pD of a manifold D consists of all tangent vectors at x_0 and is defined as

$$T_{x_0}D = \{\dot{x}(0) | V \subseteq D \text{open}, x : V \to \mathbb{R}^n, 0 \in V, x(0) = x_0, x(t) \in D, \forall t \in V, x(t) \text{ is differentiable at } t = 0\},$$

according to Definition 4.1 in [3]. Applying this definition to our tangent spaces $T_pW^{s,u}(\gamma_\epsilon(t))$ of the stable and unstable manifolds associated with $\gamma_\epsilon(t)$, we find that

$$T_pW^{s,u}(\gamma_\epsilon(t)) = \{\dot{x}(0) | x : (-\epsilon, \epsilon) \to \mathbb{R}^3, x(0) = p, x(t) \in W^{s,u}(\gamma_\epsilon(t)), \forall t \in (-\epsilon, \epsilon), x(t) \text{ is differentiable at } t = 0\}$$

As the trajectories $q^{s,u}_\epsilon(t)$ depend parametrically on t_0 and ϕ_0, we can form the derivatives of these trajectories with respect to t_0 and ϕ_0 and evaluate at $t = 0$. For ϵ sufficiently small, the points on $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ that are closest to $\gamma_\epsilon(t)$ in terms of time of flight can be parametrized by t_0 and ϕ_0. Thus, the vectors

$$\left(\frac{\partial q^s_\epsilon}{\partial t_0}, \frac{\partial q^s_\epsilon}{\partial \phi_0}\right) \tag{42}$$

and

$$\left(\frac{\partial q^u_\epsilon}{\partial t_0}, \frac{\partial q^u_\epsilon}{\partial \phi_0}\right) \tag{43}$$

form a basis for $T_pW^s(\gamma_\epsilon(t))$ and $T_pW^u(\gamma_\epsilon(t))$, respectively. To make sure that $T_pW^s(\gamma_\epsilon(t))$ and $T_pW^u(\gamma_\epsilon(t))$ are not tangent at t, one of the transversality conditions

$$\frac{\partial q^u_\epsilon}{\partial t_0} - \frac{\partial q^s_\epsilon}{\partial t_0} \neq 0 \tag{44}$$

or

$$\frac{\partial q^u_\epsilon}{\partial \phi_0} - \frac{\partial q^s_\epsilon}{\partial \phi_0} \neq 0 \tag{45}$$

is required.

To check wether these conditions are fulfilled, we differentiate $d(t_0, \phi_0, \epsilon)$ with respect to t_0 and ϕ_0 and evaluate at the intersection point $(\bar{t}_0 + \mathcal{O}(\epsilon), \bar{\phi}_0)$ where the Melnikov function attains a simple zero. Keeping expression (15) in mind, differentiation with respect to t_0 gives

$$\frac{\partial d(\bar{t}_0, \bar{\phi}_0, \epsilon)}{\partial t_0} = \frac{DH(q_0(-\bar{t}_0)) \cdot \left(\frac{\partial q^u_\epsilon}{\partial t_0} - \frac{\partial q^s_\epsilon}{\partial t_0}\right)}{\|(DH(q_0(-\bar{t}_0))\|}, \tag{46}$$

so it follows that $\frac{\partial d(\bar{t}_0, \bar{\phi}_0, \epsilon)}{\partial t_0} \neq 0$ is a sufficient condition for transversality. Considering expression (41), we find that (46) is also equal to

$$\begin{aligned} \frac{\partial d(\bar{t}_0, \bar{\phi}_0, \epsilon)}{\partial t_0} &= \epsilon \frac{\partial \tilde{d}(\bar{t}_0, \bar{\phi}_0, \epsilon)}{\partial t_0} \\ &= \frac{\epsilon}{\|(DH(q_0(-\bar{t}_0))\|} \frac{\partial M(\bar{t}_0, \bar{\phi}_0)}{\partial t_0} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{47}$$

Similarly, differentiating $d(t_0, \phi_0, \epsilon)$ with respect to ϕ_0 and evaluating at $(\bar{t}_0 + \mathcal{O}(\epsilon), \phi_0)$ gives

$$\begin{aligned}
\frac{\partial d(\bar{t}_0, \bar{\phi}_0, \epsilon)}{\partial \phi_0} &= \frac{DH(q_0(-\bar{t}_0)) \cdot \left(\frac{\partial q_\epsilon^u}{\partial \phi_0} - \frac{\partial q_\epsilon^s}{\partial \phi_0}\right)}{\|(DH(q_0(-\bar{t}_0))\|} \\
&= \epsilon \frac{\partial \tilde{d}(\bar{t}_0, \bar{\phi}_0, \epsilon)}{\partial \phi_0} \\
&= \frac{\epsilon}{\|(DH(q_0(-\bar{t}_0))\|} \frac{\partial M(\bar{t}_0, \bar{\phi}_0)}{\partial \phi_0} + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{48}$$

Thus, from (47) and (48) it is evident that when either $\frac{\partial M(\bar{t}_0, \bar{\phi}_0)}{\partial \phi_0} \neq 0$ or $\frac{\partial M(\bar{t}_0, \bar{\phi}_0)}{\partial t_0} \neq 0$, $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ intersect transversely. This completes our proof of the Theorem, as these conditions were in our assumptions.

Application of the Melnikov function to the Duffing oscillator

Let us now consider the Duffing oscillator with weak harmonic forcing and damping, which is given by the second-order differential equation

$$\ddot{x} - x + x^3 = \epsilon(\gamma \cos(\omega t) - \delta \dot{x}), \tag{49}$$

with variable parameters $\gamma, \omega, \delta > 0$ and small parameter ϵ satisfies $0 < \epsilon \ll 1$. The force amplitude serves as γ, the frequency is denoted with ω and δ represents the damping. Equation (49) is equivalent to the planar system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + \epsilon(\gamma \cos(\omega t) - \delta y), \end{cases} \tag{50}$$

where $f(x, y, t, \epsilon) = (f_1, f_2) = (0, \gamma \cos(\omega t) - \delta y)$.

By introducing $\phi = \omega t$, we obtain the autonomous vector field

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + \epsilon(\gamma \cos(\phi) - \delta y) \\ \dot{\phi} = \omega, \end{cases} \tag{51}$$

which does not explicitly depend on time.

For $\epsilon = 0$, the unforced, undampened Duffing oscillator is a conservative system and (50) reduces to to Hamiltonian system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases} \tag{52}$$

with the Hamiltonian function

$$H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}. \tag{53}$$

If we define the potential energy $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ and view $\frac{y^2}{2}$ as the kinetic energy, $H(x, y) = \frac{y^2}{2} + V(x)$ portrays the total energy of the system. For $\epsilon = 0$, all solutions of the system lie on level curves of $H(x, y)$. To determine the fixed points of (52), we set $\dot{x} = 0$ and $\dot{y} = 0$ and we obtain

$$\begin{aligned} \dot{x} = y = 0 &\implies y = 0 \\ \dot{y} = x(1 - x^2) = 0 &\implies x = 0 \lor x = \pm 1, \end{aligned}$$

so we find the three equilibria $(x, y) = (0, 0)$ and $(x, y) = (\pm 1, 0)$. To determine the stability, the Jacobian matrix of (50)

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}$$

is evaluated at the fixed points. We find

$$J(-1, 0) = J(1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_{1,2} = \pm i\sqrt{2}$, which indicates that $(x, y) = (\pm 1, 0)$ are centers. Also, we obtain

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_{1,2} = \pm 1$, thus $(x, y) = (0, 0) = p_0$ is a hyperbolic saddle point. We note that p_0 is on the level set $H(x, y) = 0$. By way of rewriting (53), we find that this level set is defined by $y^2 = x^2 - \frac{x^4}{2}$. To find the pair of homoclinic orbits at $\epsilon = 0$, we substitutie this equation into the system (52), which results in

$$\begin{aligned} y = \frac{dx}{dt} &= \pm\sqrt{x^2 - \frac{x^4}{2}} \\ \int \frac{\sqrt{2}}{|x|\sqrt{2 - x^2}} dx &= \int \pm 1 dt \\ -\tanh^{-1}\left(\sqrt{1 - \frac{x^2}{2}}\right) &= \pm t + C \\ \frac{x^2}{2} &= 1 - \tanh^2(\pm t) = \operatorname{sech}^2(t) \\ x &= \pm\sqrt{2}\operatorname{sech}(t). \end{aligned} \tag{54}$$

Differentiating $-\tanh^{-1}\left(\sqrt{1 - \frac{x^2}{2}}\right)$ verifies that our calculation is correct, as we indeed observe that

$$\begin{aligned} -\frac{d}{dx}\tanh^{-1}\left(\sqrt{1 - \frac{x^2}{2}}\right) &= \frac{-1}{1 - (1 - \frac{x^2}{2})} \frac{-x}{2\sqrt{1 - \frac{x^2}{2}}} \\ &= \frac{2x}{2x^2\sqrt{1 - \frac{x^2}{2}}} = \frac{\sqrt{2}}{x\sqrt{2 - x^2}}, \end{aligned} \tag{55}$$

which was integrated in the second computation of (54). Now, we subsitute our expression for x into (52) to find

$$\begin{aligned} \dot{y} &= \pm\sqrt{2}\operatorname{sech}(t) - (\pm\sqrt{2}\operatorname{sech}(t))^3 \\ &= \pm\sqrt{2}\operatorname{sech}(t)(1 - 2\operatorname{sech}^2(t)). \end{aligned} \tag{56}$$

By integration of (56), we find the pair of homoclinic orbits

$$q_0^{\pm}(t) = (x_0^{\pm}(t), y_0^{\pm}(t)) = (\pm\sqrt{2}\operatorname{sech}(t), \mp\sqrt{2}\operatorname{sech}(t)\tanh(t)), \tag{57}$$

shown in Figure 4.

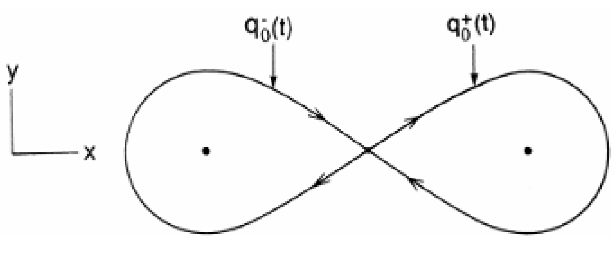

Figure 4: The two centers $(x, y) = (\pm 1, 0)$ and the hyperbolic saddle point $(x, y) = (0, 0)$ are shown, as well as the pair of homoclinic orbits $q_0^{\pm}(t) = (x_0^{\pm}(t), y_0^{\pm}(t)) = (\pm\sqrt{2}\operatorname{sech}(t), \mp\sqrt{2}\operatorname{sech}(t)\tanh(t))$ (Fig 28.4.1 from [1]).

We compute the Melnikov function for $q_0^{\pm}(t)$. Making use of (35), it follows that $M^{\pm}(t_0, \phi_0)$ is given by

$$\begin{aligned}
M^{\pm}(t_0,\phi_0) &= \int_{-\infty}^{\infty} DH(q_0^{\pm}(t)) \cdot f(q_0^{\pm}(t), \omega(t+t_0)+\phi_0, 0)dt \\
&= \int_{-\infty}^{\infty} \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} (q_0^{\pm}(t)) \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (q_0^{\pm}(t), \omega(t+t_0)+\phi_0, 0)dt \\
&= \int_{-\infty}^{\infty} \begin{pmatrix} -x+x^3 \\ y \end{pmatrix} (x_0^{\pm}(t), y_0^{\pm}(t)) \cdot \begin{pmatrix} 0 \\ \gamma\cos(\omega(t+t_0)+\phi_0) - \delta y_0^{\pm}(t) \end{pmatrix} dt \\
&= \int_{-\infty}^{\infty} y_0^{\pm}(t)(\gamma\cos(\omega(t+t_0)+\phi_0) - \delta y_0^{\pm}(t))dt \\
&= \int_{-\infty}^{\infty} \gamma y_0^{\pm}(t)\cos(\omega(t+t_0)+\phi_0) - \delta(y_0^{\pm}(t))^2 dt.
\end{aligned} \tag{58}$$

Now, as we substitute $y_0^{\pm}(t) = \mp\sqrt{2}\operatorname{sech}(t)\tanh(t)$ from (57), the Melnikov function becomes

$$\begin{aligned}
M^{\pm}(t_0,\phi_0) &= \int_{-\infty}^{\infty} (\mp\sqrt{2}\operatorname{sech}(t)\tanh(t)\gamma\cos(\omega(t+t_0)+\phi_0) - \delta(\mp\sqrt{2}\operatorname{sech}(t)\tanh(t))^2)dt \\
&= \mp\gamma\sqrt{2}\int_{-\infty}^{\infty} \operatorname{sech}(t)\tanh(t)\cos(\omega t+\omega t_0+\phi_0)dt - 2\delta\int_{-\infty}^{\infty} \operatorname{sech}(t)^2\tanh(t)^2 dt \\
&= \pm\gamma\pi\omega\sqrt{2}\operatorname{sech}\left(\frac{\pi\omega}{2}\right)\sin(\omega t_0+\phi_0) - \frac{4\delta}{3}.
\end{aligned} \tag{59}$$

Now, we consider the case $\delta = 0$ and examine the cross-section $\Sigma^{\phi_0} = \{(q,\phi) \in \mathbb{R}^2 \times S^1 | \phi = \phi_0\}$ as defined in 10 with the corresponding Poincaré map $P_\epsilon^{\phi_0}$ as described in 11. We find that the Melnikov function is has a zero, meaning the stable and the unstable manifolds intersect, for $\sin(\omega t + \phi_0) = 0 \implies \phi_0 = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ at $t = 0$. Altering the cross-section Σ^{ϕ_0} can change the symmetry properties of the Poincaré map $P_\epsilon^{\phi_0} : \Sigma^{\phi_0} \to \Sigma^{\phi_0}$ [1].

From expression (59), we find that the critical condition in terms of parameters $\delta, \gamma, \omega > 0$, for which the Melnikov function has a simple zero, causing the invariant manifolds to intersect, is given by

$$\begin{aligned}
\frac{4\delta}{3} &= \pm\sqrt{2}\gamma\pi\omega\operatorname{sech}\left(\frac{\pi\omega}{2}\right)\sin(\omega t_0+\phi_0) \\
\delta &= \frac{\pm 3\sqrt{2}\gamma\pi\omega\operatorname{sech}\left(\frac{\pi\omega}{2}\right)\sin(\omega t_0+\phi_0)}{4} \\
\left(\frac{\delta}{\gamma}\right) &\leq \frac{3\pi\omega\operatorname{sech}\left(\frac{\pi\omega}{2}\right)}{2\sqrt{2}},
\end{aligned} \tag{60}$$

as $|\sin(\omega t_0 + \phi_0)| \leq 1$ and $\delta, \gamma > 0$. This means for $\epsilon > 0$ sufficiently small, system (51) has transverse homoclinic orbits.

From (59), chaotic behavior is guaranteed for trajectories whose initial data are close enough to the homoclinic orbits $q_0^{\pm}(t)$, when

$$\frac{\delta}{\gamma} \leq \left(\frac{\delta}{\gamma}\right)_{crit} = \frac{3\pi\omega\operatorname{sech}\left(\frac{\pi\omega}{2}\right)}{2\sqrt{2}}. \tag{61}$$

In this case, $\left(\frac{\delta}{\gamma}\right)_{crit}$ is a threshold function, see Figure 5 [4].

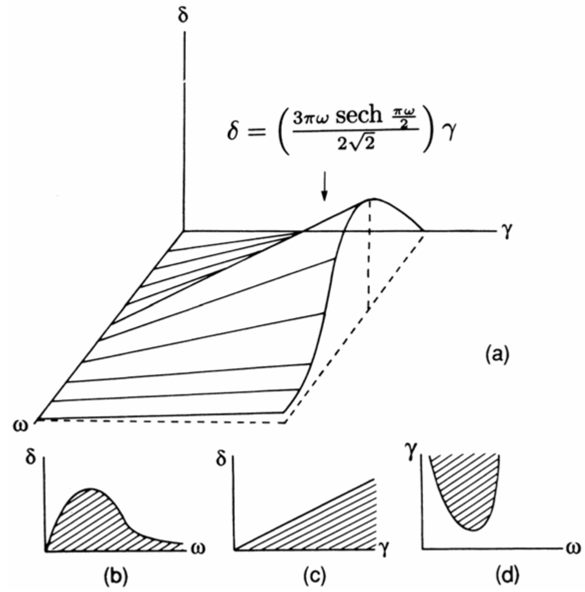

Figure 5: (a) Graph of the critical surface $\delta = \frac{3\gamma\pi\omega \operatorname{sech}\left(\frac{\pi\omega}{2}\right)}{2\sqrt{2}}$ with cross-sections where (b) γ is held constant (c) ω is held constant and (d) δ is held constant (Fig 28.5.2 from [1]).

We want to ensure that δ/γ is small enough for the manifolds to interact dynamically. Note that this criterion is independent of the specific cross-section Σ^{ϕ_0}, as it should be. Thus, (61) poses as a criterion for chaos in the Duffing oscillator with weak harmonic forcing and damping as a function of the parameters (δ, ω, γ).

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