Inleiding Niet Lineaire Dynamische Systemen

Poincaré-Bendixson Theorem

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In this paper, I will discuss the Poincaré-Bendixson Theorem. The Poincaré-Bendixson Theorem is the following (Kuznetsov, Diekmann, and Beyn, Theorem 4.13):

Theorem. *Let $x \in \mathbb{R}^2$ and assume that $\Gamma^+(x)$ belongs to a bounded closed subset of $\mathbb{R}^2$ containing only a finite number of equilibria. Then one of the following possibilities holds:*

(1) $\omega(x)$ is an equilibrium;
(2) $\omega(x)$ is a periodic orbit;
(3) $\omega(x)$ consists of equilibria and orbits having these equilibria as their α- and ω-limit sets.

In order to discuss this theorem, this paper consists of three sections. In the first section, I will discuss ω-limit sets and their main characteristics. This will be useful for section 2, in which the most important part of the paper will take place: the proof of the Poincaré-Bendixson Theorem. Lastly, in the third section, I will mention and discuss a few examples in which the Poincaré-Bendixson Theorem can be applied.

This paper is mainly based and inspired on *Dynamical Systems Essentials* written by Kuznetsov, Diekmann, and Beyn. To be more precise, I will mainly focus on Chapter 4 *Planar ODE* of the above mentioned lecture notes.

Contents

1 The ω-limit sets

For this section, I will primarily use chapter 4 *Planar ODE*, paragraph 1 *Limit sets* (Kuznetsov, Diekmann, and Beyn).

1.1 Definition

Suppose $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and $f \in C^1$. The latter means that the function f is continuous differentiable (Kuznetsov and Hanßmann). The flow that the system above generates is $\varphi(t, x)$, often written as $\varphi^t(x)$.
When $\varphi(t, x)$ is defined, the map $(t, x) \to \varphi(t, x)$ is surely continue (Kuznetsov, Diekmann, and Beyn, implied by Theorem 1.5).

The definition of an ω-limit point in combination with an ω-limit set is the following (Kuznetsov, Diekmann, and Beyn, Definition 4.1):

Definition 1. *A point $y \in \mathbb{R}^n$ is an ω-limit point of $x \in \mathbb{R}^n$, if $\varphi(t, x)$ is defined for all $t \geq 0$ and there exists an increasing sequence $\{t_i\} \to \infty$, such that $\varphi(t_i, x) \to y$ for $i \to \infty$. The set of all ω-limit points of x is called the ω-limit set of x and denoted by $\omega(x)$.*

Since this definition is quite complex, and it has a few remarks, I will explain it in more detail.
What is meant with an increasing sequence $\{t_i\} \to \infty$ when $i \to \infty$ is that the sequence $\{t_i\}$ diverges (Kuznetsov, Diekmann, and Beyn, Definition 4.1, Remark (1)). Each t_i is a 'time point', and suppose we start at a point x, the flow causes that for some t_i, we are at a different point. For simplicity, the figure below shows an example of such an increasing sequence in the case of $n = 2$. Note that $t_0 < t_1 < t_2$.

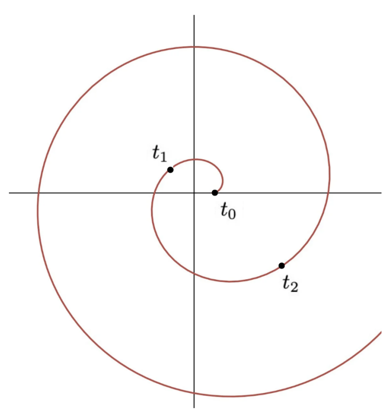

Figure 1: Example with $n = 2$

A divergent sequence is the opposite of a convergent sequence. The definition of a convergent sequence is the following (Van den Ban, Definition 3.6):

Definition. *A sequence* $(a_n)_{n\in\mathbb{N}}$ *converges if there is an element* a *such that* $a_n \to a$ *for* $n \to \infty$*. Here,* $\lim_{n\to\infty} a_n = a$ *means that for all* $\epsilon > 0$*, there exists a* $N \in \mathbb{N}$ *such that for all* $n > N$*,* $d(a_n, a) < \epsilon$*.*

Therefore, the fact that the sequence $\{t_i\}$ diverges means that there does not exist a $t \in \mathbb{R}$ such that $t_i \to t$ for $i \to \infty$.
An ω-limit set considers the long-term behavior of $x \in \mathbb{R}^n$ in time. Therefore, it is important to look at what will eventually happen, which thus implies that the sequence $\{t_i\}$ must diverge: $\{t_i\} \to \infty$ as $i \to \infty$.

In general, we denote $\Gamma = \Gamma(x)$ as the orbit through x. As noted in Definition 1, we only look at $t \geq 0$ (when $t \leq 0$ we get the α-limit set ($\alpha(x)$) instead of the ω-limit set (Kuznetsov, Diekmann, and Beyn, Definition 4.1, Remark (2))). Therefore, it is sufficient to look at the *positive half-orbit* through x, $\Gamma^+ = \Gamma^+(x)$, with $\Gamma^+(x) = \{y \in \mathbb{R}^n : y = \varphi^t(x) \text{ where } t \geq 0\}$ (Kuznetsov, Diekmann, and Beyn, Definition 4.1, Remark (3)). Note that Definition 1 implies that the ω-limit set is equal for all points on the same orbit, $\omega(\varphi^t(x)) = \omega(x)$. This means that we can talk about $\omega(\Gamma^+)$.

An equilibrium point is a point x_0 such that $\dot{x} = f(x_0) = 0$ (Kuznetsov and Hanßmann). This implies that there is no 'movement'. Therefore, when we start at x_0 and we take $\{t_i\} \to \infty$, we will stay at x_0. In other words, $\omega(x_0) = \{x_0\}$ (Kuznetsov, Diekmann, and Beyn, Definition 4.1, Remark (4)). An interesting and important addition to this is that due to the fact that at an equilibrium point there is no 'movement', $\Gamma_0 = \Gamma(x_0) = \{x_0\}$. This implies that $\omega(x_0) = \Gamma_0$.

1.2 Properties

In this section, we only focus on bounded orbits of smooth n-dimensional ODEs.

To define the basic properties of the ω-limit sets for bounded orbits of smooth n-dimensional ODEs, we first have to define the following (Kuznetsov, Diekmann, and Beyn, page 116):

Definition 2. *For any two subsets* $A, B \in \mathbb{R}^n$ *define*

$$\rho(A, B) = \inf_{x\in A, y\in B} ||x - y||$$

If A *consists of one point* $x \in \mathbb{R}^n$ *and* B *is fixed, the function* $x \mapsto \rho(x, B)$ *is continuous.*

When we assume that $x \in \mathbb{R}^n$ and that $\Gamma^+(x)$ is bounded, we can define some basic (Kuznetsov, Diekmann, and Beyn, Lemma 4.2) and some additional (Kuznetsov, Diekmann, and Beyn, Lemmas 4.3, 4.4 and 4.5) properties.

Theorem 1. *Suppose $x \in \mathbb{R}^n$ and $\Gamma^+(x)$ bounded. Properties of the ω-limit set, $\omega(x)$, are:*

(1) The set $\omega(x)$ is nonempty;
(2) The set $\omega(x)$ is bounded;
(3) The set $\omega(x)$ is closed;
(4) The set $\omega(x)$ is connected;
(5) The set $\omega(x)$ is an invariant set of the flow φ^t:
*if $y \in \omega(x)$ then $\varphi^t(y) \in \omega(x)$ for all $t \in \mathbb{R}$ (**Invariance**).*

*Above that, $\rho(\varphi(t,x), \omega(x)) \to 0$ as $t \to \infty$ (**Convergence**).*
*When we assume in addition that $r, z \in \mathbb{R}^n$, if $z \in \omega(r)$ and $r \in \omega(x)$ then $z \in \omega(x)$ (**Transitivity**).*

This theorem contains a lot of information, and therefore each claim will be discussed individually.

Remember that $\Gamma^+(x) = \{y \in \mathbb{R}^n : y = \varphi^t(x) \text{ for all } t \geq 0\}$. In Theorem 1, it is assumed that $\Gamma^+(x)$ is bounded. Therefore, we know that $\varphi^t(x)$ is defined for all $t \geq 0$.

1.2.1 The set $\omega(x)$ is nonempty

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.2, Proof (i)):
Suppose $x_i = \varphi(i, x)$ with $i = 1, 2, 3, ...$ Note that $\{x_i\}$ is an infinite sequence. Remember that $\Gamma^+(x) = \{y \in \mathbb{R}^n : y = \varphi^t(x) \text{ where } t \geq 0\}$. In this case, we consider the time points i. Therefore, x_i are points on this positive half orbit. Since $\Gamma^+(x)$ is bounded, $\{x_i\}$ is bounded as well.
The theorem below, Bolzano Weierstrass, implies an important result (Van den Ban, Theorem 4.5).

Theorem. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^p$. If the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, there is a convergent subsequence.*

The definition of a subsequence is stated below (Van den Ban, Definition 4.1).

Definition. *Suppose (V, d) is a metric space and $(a_n)_{n\in\mathbb{N}}$ a sequence in V. A subsequence of (a_n) is a sequence such as $(a_{n_k})_{k\in\mathbb{N}}$ with $(n_k)_{k\in\mathbb{N}}$ a strictly monotone increasing sequence in $\mathbb{N}$; $n_k < n_{k+1}$ for all $k \geq 0$.*

From the Bolzano-Weierstrass Theorem, we can conclude that $\{x_{i_l}\} \to m$ ($l \to \infty, l \in \mathbb{N}$) (convergent subsequence). By definition we have that $\{i_l\} \to \infty$ when $l \to \infty$. So, there exists an increasing sequence $\{i_l\} \to \infty$ such that $\varphi(i_l, x) \to m$ for $l \to \infty$. Therefore, from Definition 1 we can conclude that $m \in \omega(x)$ which implies that $\omega(x) \neq \emptyset$. □

1.2.2 The set $\omega(x)$ is bounded

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.2, Proof (ii)):
This claim will be proved by contradiction. So, assume that $\omega(x)$ is unbounded.

From Definition 1 follows that when we define $m_i = \varphi(t_i, x)$, there exists an increasing sequence $\{t_i\} \to \infty$, such that $m_i \to \infty$ as $i \to \infty$. Remember $\Gamma^+(x) = \{y \in \mathbb{R}^n : \varphi^t(x) = y \text{ where } t \geq 0\}$. Therefore, this implies that $\Gamma^+(x)$ is unbounded, which is a contradiction. We can conclude that $\omega(x)$ is bounded. □

1.2.3 The set $\omega(x)$ is closed

Below is the definition of a closed set (Van den Ban, Definition 2.7).

Definition. *A set $A \in \mathbb{R}^n$ is closed if each limit point of A belongs to A.*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.2, Proof (iii)):
Suppose that (arbitrary) $\{m_i\}$ is a sequence of points $m_i \in \omega(x)$. Assume that $\{m_i\} \to m \in \mathbb{R}^n$. Note that we thus have $\lim_{i\to\infty} m_i = m$ which means that m is a limit point of $\omega(x)$. To prove that $\omega(x)$ is closed, it is, by the definition stated at the beginning of this section, sufficient to prove that $m \in \omega(x)$ (since $m_i \in \omega(x)$ arbitrary).

Since $m_i \in \omega(x)$, we know from Definition 1 that there exists an increasing sequence of times $\{t_l^{(i)}\} \to \infty$ for $l \to \infty$, such that when we define $x_l^{(i)} = \varphi(t_l^{(i)}, x)$, we have that $\{x_l^{(i)}\} \to m_i$ for $l \to \infty$.
In other words, we have that $||x_l^{(i)} - m_i|| \to 0$ if $l \to \infty$.

The above therefore states that the bigger l ($l \to \infty$), the smaller $||x_l^{(i)} - m_i||$ ($||x_l^{(i)} - m_i|| \to 0$). Therefore, we can take the first $L(i)$, for each i, with $||x_{L(i)}^{(i)} - m_i|| \leq \frac{1}{i}$. If we look at $||x_{L(i)}^{(i)} - m||$, we know that this can be rewritten as $||(x_{L(i)}^{(i)} - m_i) + (m_i - m)||$. We defined $L(i)$ in such a way that $||x_{L(i)}^{(i)} - m_i|| \leq \frac{1}{i}$, so by the triangle inequality we have that $||x_{L(i)}^{(i)} - m|| \leq \frac{1}{i} + ||m_i - m||$. Note that when we take $i \to \infty$, $\frac{1}{i} + ||m_i - m|| \to 0$. Remember that $||x_{L(i)}^{(i)} - m|| \geq 0$. Therefore, when $i \to \infty$, $||x_{L(i)}^{(i)} - m|| \to 0$. This means that $\{x_{L(i)}^{(i)}\} \to m$ for $i \to \infty$. Remember that $x_{L(i)}^{(i)} = \varphi(t_{L(i)}^{(i)}, x)$, and $\{t_{L(i)}^{(i)}\} \to \infty$ for $i \to \infty$.
By Definition 1, we know that $m \in \omega(x)$. □

1.2.4 The set $\omega(x)$ is connected

Below, I will prove the claim from Theorem 1 that under a few assumptions, the set $\omega(x)$ is connected. However, first I will state the definition of a connected set (Bashirov, Paragraph 4.8 Connectedness):

Definition. *The set A is connected if and only if the only subsets of A that are both open and closed are A itself and $\emptyset$.*

The intuition of this definition is that a connected set cannot be divided in two disjoint open subsets B and C (both nonempty) such that $A = B \cup C$.

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.2, Proof (iv)):
The claim will be proved by contradiction, so suppose that $\omega(x)$ is disconnected. This implies that $\omega(x) = M \cup N$, for $M, N \subset \mathbb{R}^n$ with $M, N \neq \emptyset$. From Theorem 1, we know that $\omega(x)$ is a bounded set, which therefore implies that the subsets M, N are bounded as well. Above that, as mentioned at the beginning of this section, M and N are disjoint sets.

As proved in section 1.2.3, we know that $\omega(x)$ is closed. This implies that M and N are also closed subsets of $\mathbb{R}^n$ (Van den Ban, Lemma 2.29). Note that since M and N are disjoint sets, they have no elements in common. This implies that $\rho(M, N) = c > 0$.
Note that $\omega(x) = M \cup N$, which implies that both M and N are composed of ω-limit points. This implies that there exist sequences $\{t_i^M\}, \{t_i^N\}$, such that $\varphi(t_i^M, x), \varphi(t_i^N, x)$ get very close to the sets M and N, respectively. In other words, $\rho(M, \varphi(t_i^M, x)) < \frac{c}{2}$ and $\rho(N, \varphi(t_i^N, x)) < \frac{c}{2}$. Due to the fact that $\rho(M, N) = c$, this implies that $\rho(M, \varphi(t_i^N, x)) > \frac{c}{2}$.
Now, suppose that $t_i^M < t_i^N$ and $t \in [t_i^M, t_i^N]$. Consider the continuous function (since ρ, φ continuous) $f(t) = \rho(M, \varphi(t, x))$. The intermediate value theorem (Van den Ban, Theorem 3.46) now implies that there exists some $t_i \in [t_i^M, t_i^N]$ such that $f(t_i) = \frac{c}{2}$, and therefore there exists some sequence $\{t_i\}$ for which the following holds: $\rho(M, \varphi(t_i, x)) = \frac{c}{2}$. Note that $\{\varphi(t_i, x)\}$ is bounded. By the theorem (about the convergent subsequence) stated in section 1.2.1, we now have $\{\varphi(t_{i_l}, x)\} \to m$ as $l \to \infty$. This implies that $m \in \omega(x)$. However, since $\rho(M, \varphi(t_i, x)) = \frac{c}{2} > 0$, so $m \notin M$. Similarly, due to that $\rho(M, \varphi(t_i^N, x)) > \frac{c}{2}$ and $\rho(M, \varphi(t_i, x)) = \frac{c}{2}$, we also have that $m \notin N$. However, we stated that $\omega(x) = M \cup N$. In other words, we have proved that $m \in \omega(x)$, but $m \notin M \cup N$. Therefore, we have a contradiction. □

1.2.5 Invariance

Remember that in Theorem 1, the following claim was made:

Theorem. *Suppose $x \in \mathbb{R}^n$ and $\Gamma^+(x)$ bounded. The set $\omega(x)$ is an invariant set of the flow φ^t: if $y \in \omega(x)$ then $\varphi^t(y) \in \omega(x)$ for all $t \in \mathbb{R}$*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.4, Proof):
This proof will consist of two parts: the proof for forward invariance, and the proof in the case of backward invariance.

Part 1: Forward invariance
Suppose $y \in \omega(x)$. As this proof considers forward invariance, we have to prove that $\varphi(t, y) \in \omega(x)$ for $t > 0$.
Since $y \in \omega(x)$, we know by Definition 1 that there exists an increasing sequence $\{t_i\} \to \infty$ such that $\varphi(t_i, x) \to y$ for $i \to \infty$. Suppose, similar to, for example, the proof in section 1.2.1, that $x_i = \varphi(t_i, x)$, such that we have $x_i \to y$ as $i \to \infty$.

Now, consider the sequence $m_i = \varphi(t, x_i)$. Note that we defined $x_i = \varphi(t_i, x)$ so we can write $m_i = \varphi(t, \varphi(t_i, x))$. From here, we can conclude two things:

- We know that in general $\varphi^a \circ \varphi^b = \varphi^{a+b}$ (Kuznetsov and Hanßmann, Paragraph 3.3), which implies that $m_i = \varphi(t + t_i, x)$.
- Note that $\varphi(\cdot, \cdot)$ is a continuous function and that we assumed that $x_i \to y$ as $i \to \infty$. Therefore, $\varphi(t, \varphi(t_i, x)) \to \varphi(t, y)$ as $i \to \infty$. We can therefore say that $\{m_i\} \to \varphi(t, y)$ as $i \to \infty$.

These two results combined therefore imply that for an increasing sequence $\{t + t_i\} \to \infty$, $\varphi(t + t_i, x) \to \varphi(t, y)$ as $i \to \infty$. Note that from Definition 1, it now follows that $\varphi(t, y) \in \omega(x)$. □

Part 2: Backward invariance
Suppose $y \in \omega(x)$. As this proof considers backward invariance, we have to prove that $\varphi^t(y) \in \omega(x)$ for $t < 0$.
The same steps as in part 1 can be used, however we have to discuss a specific case which will be mentioned in the proof.
Since $y \in \omega(x)$, we know by Definition 1 that there exists an increasing sequence $\{t_i\} \to \infty$ such that $\varphi(t_i, x) \to y$ for $i \to \infty$. Suppose that $x_i = \varphi(t_i, x)$, such that we have $x_i \to y$ as $i \to \infty$.
Now, consider the sequence $n_i = \varphi^t(x_i) = \varphi(t, \varphi(t_i, x))$, with $t < 0$ (*). Similar to what we did in part 1, we can conclude that for an increasing sequence $\{t + t_i\} \to \infty$ as $i \to \infty$, $\varphi(t + t_i, x) \to \varphi(t, y)$ as $i \to \infty$. This means that (Definition 1), $\varphi(t, y) \in \omega(x)$. □

* There is the possibility that $\varphi^t(x_i)$ is not defined for $i \leq N$ but $\varphi^t(x_i)$ is defined for $i > N$, for some $N > 0$ ($N \in \mathbb{N}$), since $t + t_i$ is too small (due to the fact that $t < 0$). When this is the case, it can be solved by considering $t + t_{N+i}$, since $\varphi(t + t_{N+i}, x)$ is defined.

1.2.6 Convergence

Remember that in Theorem 1, the following claim was made:

Theorem. *Suppose $x \in \mathbb{R}^n$ and $\Gamma^+(x)$ bounded. Then $\rho(\varphi(t, x), \omega(x)) \to 0$ as $t \to \infty$.*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.3, Proof):
This claim will be proved by contradiction. Therefore, assume that there exists an $c > 0$ and a sequence $\{t_i\} \to \infty$ such that, for all $i = 1, 2, 3, \ldots$, $\rho(\varphi(t_i, x), \omega(x)) \geq c > 0$.
We will now use a similar method as in section 1.2.1: suppose $x_i = \varphi(t_i, x)$. By similar reasoning as in section 1.2.1, we know that $\{x_i\}$ is a bounded and infinite sequence.
By the theorem stated in section 1.2.1, we know that there exists a convergent subsequence (Bolzano Weierstrass): $\{x_{i_l}\} \to m$ with $l \in \mathbb{N}$.

Since $\{t_{i_l}\} \to \infty$ when $l \to \infty$, there exists an increasing sequence $\{t_{i_l}\} \to \infty$ such that $\varphi(t_{i_l}, x) \to m$ for $l \to \infty$. This implies (Definition 1) that $m \in \omega(x)$. Note that $\rho(\varphi(t_{i_l}, x), \omega(x)) \geq c > 0$. However, just as what we did in section 1.2.5, since ρ, φ are continuous, we know that when $l \to \infty$, $\rho(\varphi(t_{i_l}, x), \omega(x)) \to \rho(m, \omega(x))$. Therefore, $\rho(m, \omega(x)) \geq c > 0$.
Since $m \in \omega(x)$, we should get $\rho(m, \omega(x)) = 0$ (since $\inf_{m,y \in \omega(x)} ||m - y|| = 0$, with $y = m$) , which implies that we have a contradiction. Therefore, $\rho(\varphi(t, x), \omega(x)) \to 0$ as $t \to \infty$. □

1.2.7 Transitivity

Remember that in Theorem 2, the following claim was made:

Theorem. *Suppose $x \in \mathbb{R}^n$ and $\Gamma^+(x)$ bounded. Assume in addition that $r, z \in \mathbb{R}^n$, if $z \in \omega(r)$ and $r \in \omega(x)$ then $z \in \omega(x)$.*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.5, Proof):
Define $x_i = \varphi(t_i, x)$. Suppose that, for an increasing sequence $\{t_i\} \to \infty$, $\{x_i\} \to r$. Note that this means $r \in \omega(x)$
Define $r_i = \varphi(\tau_i, r)$. Similarly, suppose that, for an increasing sequence $\{\tau_i\} \to \infty$, $\{r_i\} \to z$. Note that this means $z \in \omega(r)$.
Define $z_i = \varphi(\tau_i, x_i)$. Note that $x_i = \varphi(t_i, x)$, so $z_i = \varphi(\tau_i, \varphi(t_i, x))$. Again, using similar steps as in section 1.2.5, we can conclude two things:

- We know that in general, $\varphi^a \circ \varphi^b = \varphi^{a+b}$ (Kuznetsov and Hanßmann, Paragraph 3.3). Therefore, $z_i = \varphi(\tau_i + t_i, x)$.

- Note that $\varphi(\cdot, \cdot)$ is a continuous function and we assumed that $\{x_i\} \to r, \{r_i\} \to z$. This implies that when $i \to \infty$, $\varphi(\tau_i, \varphi(t_i, x)) \to z$.

When we combine these two results, we know that $\varphi(\tau_i + t_i, x) \to z$ for an increasing sequence of times $\{\tau_i + t_i\} \to \infty$ as $i \to \infty$.
By Definition 1, we have that $z \in \omega(x)$. □

2 Proof of the Poincaré-Bendixson Theorem in $\mathbb{R}^2$

For this section, I will primarily use chapter 4 *Planar ODE*, paragraph 2 *The Poincaré-Bendixson Theorem* (Kuznetsov, Diekmann, and Beyn).
We will now focus on the Poincaré-Bendixson Theorem in $\mathbb{R}^2$. Therefore, consider the following (standard) system:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \tag{1}$$

with $f, g \in C^1$.

As already written in the introduction, below is the Poincaré-Bendixson Theorem (in $\mathbb{R}^2$):

Theorem. *Let $x \in \mathbb{R}^2$ and assume that $\Gamma^+(x)$ belongs to a bounded closed subset of $\mathbb{R}^2$ containing only a finite number of equilibria. Then one of the following possibilities holds:*
(1) $\omega(x)$ is an equilibrium;
(2) $\omega(x)$ is a periodic orbit;
(3) $\omega(x)$ consists of equilibria and orbits having these equilibria as their α- and ω-limit sets.

This section will be divided in two parts. First, I will discuss all relevant information (Definitions, Lemmas and Theorems) which I will use in the second part to prove the Poincaré-Bendixson Theorem.

2.1 Relevant information

When we consider the ω-limit sets of system (1), from Definition 1 we can conclude that among those are equilibria and cycles (Kuznetsov, Diekmann, and Beyn, Paragraph 4.2). In section 1.1, equilibrium points are already discussed. The definition of a cycle is stated below (Kuznetsov, Diekmann, and Beyn, page 119):

Definition. *A closed orbit Γ_0 generated by a periodic solution (non-equilibrium and $\varphi(t,x) = \varphi(t+T,x)$ for some $T > 0$) is called a cycle. The minimal T for which this holds is referred to as the period of Γ_0.*

For the relevant lemmas to prove the Poincaré-Bendixson Theorem, a few definitions are necessary. Below is the definition of a transverse segment (Kuznetsov, Diekmann, and Beyn, Definition 4.6).

Definition 3. *A closed line segment L is called transverse for the system (1) if $(f(x,y), g(x,y)) \neq 0$ for all $(x,y) \in L$, and the vector field is nowhere tangent to L.*

The definition of a Jordan curve is the following (Kuznetsov, Diekmann, and Beyn, Definition 4.7):

Definition 4. *A continuous image of a circle without self-intersections is called a Jordan curve.*

There is a famous theorem regarding this Jordan curve (Kuznetsov, Diekmann, and Beyn, page 119):

Theorem. *The complement to a Jordan curve* $\Gamma \subset \mathbb{R}^2$ *is the union of two disjoint connected open sets, the interior* $Int(\Gamma)$ *and the exterior* $Ext(\Gamma)$*. The interior is bounded, and the exterior is unbounded*

Now we can formulate some lemmas that will be helpful in section 2.2.
The lemma below is about monotonicity (Kuznetsov, Diekmann, and Beyn, Lemma 4.8).

Lemma 1. *If an orbit of the system (1) intersects L for an increasing sequence of times* $\{t_i\}$*, then the corresponding sequence of intersection points* $\{p_i\}$ *is either constant or strictly monotone.*

Before I proof this lemma, I will give the definition of a constant (Glen) and a strictly monotone (Simon Fraser University, Definition 6.16) sequence.

Definition. *A constant sequence is a sequence where all elements of the sequence are equal to the constant value* $c \in \mathbb{R}$*.*
A strictly monotone sequence is a sequence $\{p_i\}$ *such that* $p_i < p_{i+1}$ *for all* i *(increasing,* $p_i > p_{i+1}$ *when decreasing).*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.8, Proof):
Suppose we consider the orbit that crosses a transverse segment L at $p_0 = \varphi(t_0, x)$ and $p_1 = \varphi(t_1, x)$ with $p_0 \neq p_1$. Here, $t_0 < t_1$. Assume that between these two time points, there are no other intersections.
Suppose we look at the closed curve Γ, which is composed as shown in the Figure below.

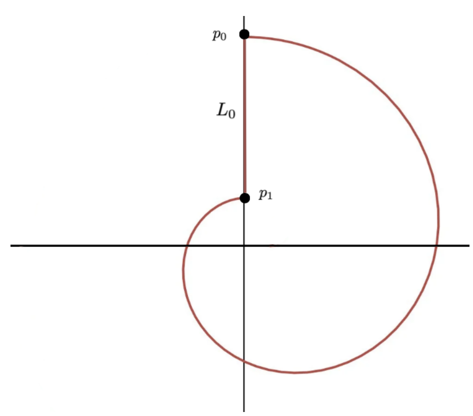

Figure 2: The figure of Γ

Note that Γ is a continuous circle without self-intersections, and therefore is a Jordan curve. Orbits cannot cross each other. Therefore, when an orbit starts in $Int(\Gamma)$, it will never leave. This implies that $Int(\Gamma)$ is forward invariant.
Since $Int(\Gamma)$ is forward invariant, and $p_1 \in Int(\Gamma)$, we can conclude that $\varphi(t, p_1) \in Int(\Gamma)$ for all $t \geq 0$. Suppose p_2 is the next intersection of the orbit with the transverse segment L: $p_2 = \varphi(t_2, x)$. Note that it is therefore the case that $p_2 \in Int(\Gamma)$.
This is true for all future intersections which implies that the sequence of intersection points $\{p_i\}$ is monotone.
The Figure below shows the idea (Kuznetsov, Diekmann, and Beyn, Figure 4.3).

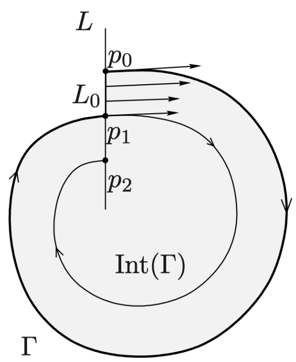

Figure 3: Proof of monotone sequence idea

Suppose that $p_0 = \varphi(t_0, x) = \varphi(t_1, x) = p_1$. In that case, we can take the closed curve from p_0 to p_1, which is a periodic orbit. This implies that the next intersection with L, the transverse segment through $p_0 = p_1$, will be at the same point: $p_0 = p_1 = p_2$. This will also be true for all future intersections. Thus $p_0 = p_1 = \ldots$. This means that $\{p_i\}$, the sequence of intersection points, is constant. □

The Lemma below will also be useful (Kuznetsov, Diekmann, and Beyn, Lemma 4.9):

Lemma 2. *Let $p \in \omega(x)$ and let L be a transverse segment through p. Then there is an increasing sequence $\{t_i\} \to \infty$ such that for $q_i = \varphi(t_i, x)$ we have $\{q_i\} \to p$ and $q_i \in L$.*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.9, Proof)
Suppose $m \in \omega(x)$ and let L be a transverse segment through m. We know from Definition 1 that there exists an increasing sequence $\{t_i\} \to \infty$ such that $\varphi(t_i, x) \to m$ for $i \to \infty$. Suppose $m_i = \varphi(t_i, x)$ such that $\{m_i\} \to m$. Note that each m_i lies on the orbit $\Gamma(x)$.

Suppose we now take a look at the points m_i that are close to m. Let $n_i = \varphi(\tau(m_i), m_i)$ with $\tau(m_i)$ the minimal time needed to shift along the orbit $\Gamma(x)$ from m_i to L. Note that this implies that $n_i \in L$. Since L is the transverse segment through m, we know that when $\{m_i\} \to m$ as $i \to \infty$, we have $\tau(m_i) \to 0$. This implies that $\rho(n_i, m_i) \to 0$ since $n_i \to \varphi(0, m_i) = m_i$ when $\tau(m_i) \to 0$.

Now considering $\rho(n_i, m)$: $||n_i - m|| = ||n_i - m_i + m_i - m||$. Therefore, by the triangle inequality $||n_i - m|| \leq ||n_i - m_i|| + ||m_i - m||$. Note that we stated $\rho(n_i, m_i) \to 0$. Similarly, we stated that $\{m_i\} \to m$. Therefore, when $i \to \infty$, $||n_i - m|| \to 0$. This implies that, when $i \to \infty$, $\{n_i\} \to m$. □

The reason such a $\tau(x)$ exists is explained in the proof of Lemma 4.9 of the lecture notes written by Kuznetsov, Diekmann, and Beyn. In this explanation they use the Implicit Function Theorem (McGill University, Theorem 1).

The two previous lemmas lead to a new result (Kuznetsov, Diekmann, and Beyn, Lemma 4.10):

Lemma 3. *The set $\omega(x)$ can intersect the transverse segment L in at most one point.*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.10, Proof):
Suppose we take the elements p and m ($p \neq m$) part of the intersection of the ω-limit set and the transverse segment L through p and m, $p, m \in \omega(x) \cap L$ (so more than one intersection point).
Note that this is exactly what is stated in the first sentence of Lemma 2. Therefore, there exist increasing sequences $\{t_i\} \to \infty$ and $\{s_i\} \to \infty$ such that for $q_i = \varphi(t_i, x)$ we have $\{q_i\} \to p$ and $q_i \in L$, and similarly for $l_i = \varphi(s_i, x)$ we have $\{l_i\} \to m$ and $l_i \in L$.
We can now formulate the orbit Γ through these points q_i and l_i. So, Γ intersects L for an increasing sequence of times $\{\tau_i\} \to \infty$ (where τ_i consists of both s_i and t_i). From Lemma 1 now follows that the sequence of intersection points $\{f_i\}$ (consists of both q_i, l_i) is either constant or strictly monotone.

When $\{f_i\}$ is constant we know that all elements of the sequence are equal to the constant value $c \in \mathbb{R}$. This therefore implies that $q_i = c, l_i = c$, but since we had $\{q_i\} \to p, \{l_i\} \to m$ it must be the case that $p = m(= c)$. Therefore, the set $\omega(x) \cap L$ consists of only one point.
Suppose that $\{f_i\}$ is strictly monotone. Note that a monotone sequence of these intersection points ($f_i \in \omega(x) \cap L$) can have at maximum one limit point, it must be the case that $p = m$. This again implies that the intersection of $\omega(x)$ and L consists of (at most) one point. □

The previous lemma, Lemma 3, therefore shows that $\omega(x) \cap L = \{m\}$ (with $\{m\}$ a single point) or $\omega(x) \cap L = \emptyset$. Another lemma, the lemma below, is used to verify the presence of a periodic orbit (Kuznetsov, Diekmann, and Beyn, Lemma 4.11).

Lemma 4. *If $\omega(x)$ is nonempty and does not contain equilibria, then it contains a periodic orbit Γ_0.*

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.11, Proof):
Suppose $\omega(x) \neq \emptyset$ and that $\omega(x)$ does not contain any equilibrium points. This implies that we can take $m \in \omega(x)$. When looking at the ω-limit set of m, we can look at $l \in \omega(m)$. We know from the theorem discussed in section 1.2.7 (Theorem 1 (Transitivity)), that $l \in \omega(x)$. Since we assumed that $\omega(x)$ contains no equilibria, $l \in \omega(x)$ is a non equilibrium point.

When we consider a transverse segment L through l, we know from Lemma 2 that there is an increasing sequence $\{t_i\}$ such that for $m_i = \varphi(t_i, m)$ we have $\{m_i\} \to l$ and $m_i \in L$.
Now Theorem 1 (5) implies that since $m \in \omega(x)$, $m_i = \varphi(t_i, m) \in \omega(x)$ for all t_i. Note that $\varphi(t_i, m) \in \Gamma(m)$, so $m_i \in L \cap \Gamma(m)$, $m_i \in \omega(x)$.

So, there is an orbit $\Gamma(m)$ that intersects L for an increasing sequence of times $\{t_i\}$, and Lemma 1 thus implies that $\{m_i\}$ is either constant or strictly monotone.
Suppose that $\{m_i\}$ is strictly monotone. From the definition we know that this implies that $m_i \neq m_j$ if $i \neq j$. Since $m_i \in L, m_i \in \omega(x)$ this implies that the intersection of $\omega(x)$ and L contains more than one point, which is a contradiction to Lemma 3.
Therefore, we know that $\{m_i\}$ is constant: $m_i = l$. Note that $m_i = \varphi(t_i, m) = l$ for some increasing sequence $\{t_i\}$.
From the definition stated at the beginning of section 2.1, we can conclude that $\Gamma(m)$ is a periodic orbit. Note that Theorem 1 implies that $\Gamma(m) \subset \omega(x)$, so $\omega(x)$ contains a periodic orbit. □

Now, the last lemma that will be discussed in this section completes the necessary information to prove the Poincaré-Bendixson Theorem, and it really is a continuation of Lemma 4 (Kuznetsov, Diekmann, and Beyn, Lemma 4.12):

Lemma 5. *If $\omega(x)$ contains a periodic orbit Γ_0, then $\omega(x) = \Gamma_0$.*

This lemma therefore states that when the ω-limit set contains a periodic orbit, than that periodic orbit is the ω-limit set (the ω-limit set does not contain anything else).

Proof (Kuznetsov, Diekmann, and Beyn, Lemma 4.12, Proof):
Suppose that $\omega(x)$ contains a periodic orbit Γ_0. Suppose that $m \in \omega(x) \cap \Gamma_0$ and that there is a transverse segment L through m (arbitrary). Since Lemma 3 implies that the intersection of the ω-limit set and a transverse segment can only contain one point (at most), $\omega(x) \cap L = \{m\}$, we know that there exists an open annulus around Γ_0 that does not contain any other points of $\omega(x)$. So, in this open annulus, the only subset of $\omega(x)$ is Γ_0.

From Theorem 1 can be concluded that $\omega(x)$ is connected. This implies that since Γ_0 is the only subset of $\omega(x)$ in this open annulus, we have $\omega(x) \setminus \Gamma_0 = \emptyset$. So, $\omega(x) = \Gamma_0$. □

2.2 Proof

Below is, again, the Poincaré-Bendixson Theorem stated:

Theorem. *Let $x \in \mathbb{R}^2$ and assume that $\Gamma^+(x)$ belongs to a bounded closed subset of $\mathbb{R}^2$ containing only a finite number of equilibria. Then one of the following possibilities holds:*
(1) $\omega(x)$ is an equilibrium;
(2) $\omega(x)$ is a periodic orbit;
(3) $\omega(x)$ consists of equilibria and orbits having these equilibria as their α- and ω-limit sets.

This Theorem will now be proved with the information discussed primarily in section 2.1 (and Chapter 1).

Proof of the Poincaré-Bendixson Theorem (Kuznetsov, Diekmann, and Beyn, Theorem 4.13, Proof):
Note that when we assume that $x \in \mathbb{R}^2$ and that $\Gamma^+(x)$ belongs to a bounded closed subset of $\mathbb{R}^2$ containing only a finite number of equilibria, there are three possibilities for $\omega(x)$:

- $\omega(x)$ contains only equilibria.
- $\omega(x)$ contains no equilibria.
- $\omega(x)$ contains both equilibrium and non-equilibrium points.

Suppose that $\omega(x)$ contains only equilibria. This implies that $\omega(x)$ is the union of a finite number of equilibrium points. Theorem 1 implies that $\omega(x)$ is connected. The latter means that the ω-limit set cannot be divided in disjoint sets. Therefore, $\omega(x)$ contains only one equilibrium. Since it only contains equilibria, we know that $\omega(x)$ is an equilibrium.

Suppose that $\omega(x)$ contains no equilibria. We know from Theorem 1 that $\omega(x)$ is nonempty. Lemma 4 implies that $\omega(x)$ contains a periodic orbit Γ_0. Now, Lemma 5 implies that when $\omega(x)$ contains a periodic orbit Γ_0, $\omega(x) = \Gamma_0$. So, $\omega(x)$ is a periodic orbit.

Suppose that $\omega(x)$ contains both equilibrium and non-equilibrium points. Take $m \in \omega(x)$, an arbitrary non-equilibrium point. The following theorem implies an important result, but first we have to formulate the definition of an α-limit set:

Definition. *A point $y \in \mathbb{R}^n$ is an α-limit point of $x \in \mathbb{R}^n$ if $\varphi(s, x)$ is defined for all $s \leq 0$ and there exists a decreasing sequence $\{s_i\} \to -\infty$ such that*

$\varphi(s_i, x) \to y$ for $i \to \infty$. The set of all α-limit points of x is called the α-limit set of x and denoted by $\alpha(x)$.

Now we can formulate the theorem.

Theorem. *Let $x \in \mathbb{R}^2$ and assume that $\Gamma^+(x)$ belongs to a bounded closed subset of $\mathbb{R}^2$. Suppose $m \in \omega(x)$ is a non-equilibrium point. Then $\alpha(m) \subset \omega(x)$.*

Proof:
Suppose $m \in \omega(x)$, a non equilibrium point. By Theorem 1 we know that for all $t \in \mathbb{R}$, $\varphi^t(m) \in \omega(x)$. This implies that the whole orbit that goes through m lies in $\omega(x)$. Above that, Theorem 1 implies that $\omega(x)$ is closed, and thus contains all limit points. Since $\alpha(m)$ consists of all α-limit points of m, $\alpha(m) \subset \omega(x)$. $\square$

The above theorem together with Theorem 1 implies that $\omega(m), \alpha(m) \subset \omega(x)$. Again, there are now a few possibilities for $\omega(m)$:

- $\omega(m)$ does not contain an equilibrium.
- $\omega(m)$ contains both equilibrium and non-equilibrium points.
- $\omega(m)$ only contains equilibria.

Suppose $\omega(m)$ does not contain an equilibrium. As already proved above, this implies that $\omega(m)$ is a periodic orbit. Since $\omega(m) \subset \omega(x)$, this implies that $\omega(x)$ contains a periodic orbit, which would mean that $\omega(x)$ is this periodic orbit. However, this would imply that $\omega(x)$ does not contain equilibrium points, which is a contradiction.

Suppose $\omega(m)$ contains both equilibrium and non-equilibrium points. Let $l \in \omega(m)$ be a non-equilibrium point. Consider the transverse segment L through l. Theorem 1 implies that $\Gamma(m) \subset \omega(x)$. Lemma 2 implies that there is an increasing sequence $\{t_i\} \to \infty$ such that for $m_i = \varphi(t_i, m)$ we have $\{m_i\} \to l$ and $m_i \in L$. Note that $m_i \in \omega(x)$. So, $\Gamma(m)$ intersects L (more than once), and from Lemma 1 can be concluded that $\{m_i\}$ is strictly monotone or constant.
When $\{m_i\}$ is constant, $m_i = \varphi(t_i, m) = l$, which would imply that $\Gamma(m)$ is a periodic orbit. When $\Gamma(m)$ is a periodic orbit, $\Gamma(m) \subset \omega(x)$, this leads to a contradiction as already proved before.
When $\{m_i\}$ is strictly monotone, $m_i \neq m_j$ for all $i \neq j$ which would imply that the intersection $\omega(x) \cap L$ contains more than one point. The latter is a contradiction to Lemma 3.

The above two cases are thus not possible. Therefore, we know that $\omega(m)$ only contains equilibria. Using similar steps as in the proof above, this implies that $\omega(m)$ is an equilibrium.

Similar steps can be applied on $\alpha(m)$. However, Theorem 1 is about ω-limit sets and therefore does not automatically apply to α-limit sets. Therefore, below is a similar theorem about α limit sets.

Theorem 2. *Suppose $x \in \mathbb{R}^n$ and $\Gamma^-(x) = \{y \in \mathbb{R}^n : y = \varphi^s(x),\ s \leq 0\}$ is bounded. Properties of the α-limit set, $\alpha(x)$, are:*
(1) The set $\alpha(x)$ is nonempty;
(2) The set $\alpha(x)$ is bounded;
(3) The set $\alpha(x)$ is closed;
(4) The set $\alpha(x)$ is connected;
(5) The set $\alpha(x)$ is an invariant set of the flow φ^s:
*if $y \in \alpha(x)$ then $\varphi^s(y) \in \alpha(x)$ for all $s \in \mathbb{R}$ (**Invariance**).*
*Above that, $\rho(\varphi(s,x), \alpha(x)) \to 0$ as $s \to -\infty$ (**Convergence**).*
*When we assume in addition that $r, z \in \mathbb{R}^n$, if $z \in \alpha(r)$ and $r \in \alpha(x)$ then $z \in \alpha(x)$ (**Transitivity**).*

The proof of this theorem follows directly from (the proof of) Theorem 1, when we consider $s = -t$ (time-reversed). So, there are a few possibilities for $\alpha(m)$:

- $\alpha(m)$ does not contain an equilibrium.
- $\alpha(m)$ contains both equilibrium and non-equilibrium points.
- $\alpha(m)$ only contains equilibria.

Similar to the cases for $\omega(m)$, but now reformulated for α-limit sets, we can conclude that the first two options lead to a contradiction, which implies that $\alpha(m)$ only contains equilibria. From Theorem 2 can be concluded that the α-limit set is connected, and therefore $\alpha(m)$ contains only one equilibrium. Thus, we can conclude that $\alpha(m)$ is an equilibrium.

Note thus that for a non-equilibrium point $m \in \omega(x)$, both the ω-limit set and the α-limit set is an equilibrium. Thus, $\omega(x)$ consists of equilibria and orbits having these equilibria as their α- and ω-limit sets. □

3 Examples

In this section, I will discuss two examples in which we can apply the Poincaré-Bendixson Theorem. As will be seen in the examples, often the application of the Poincaré-Bendixson Theorem is about 'eliminating' options, such that there remains one option.

3.1 Example 1

Suppose we have the following system in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -y - x^2 \end{cases}$$

Suppose $l \in \mathbb{R}^2$ with $l = (x, y)$.
First, we are going to compute the equilibria of the system.
Note that $\dot{x} = 0$ if $x(-1 + y) = 0$. This is the case when $x = 0$ or $y = 1$. When $x = 0$, $\dot{y} = 0$ if $y = 0$. Similarly, if $y = 1$, $\dot{y} = 0$ if $x^2 = -1$, which leads to $x = \pm i$. Therefore, the only equilibrium in $\mathbb{R}^2$ is $(0, 0)$. We have now met the condition of having only a finite number of equilibria.
We can also find the eigenvalues of this equilibrium. Note that the Jacobian of the system is:

$$\begin{pmatrix} -1 + y & x \\ -2x & -1 \end{pmatrix}$$

At $(0, 0)$, this is equal to

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

To find the eigenvalues, we have to solve: $\det \begin{vmatrix} -1 - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0$. This implies that $(-1 - \lambda)^2 = 0$, so $\lambda = -1$. This implies that $(0, 0)$ is a stable double critical node (Kuznetsov and Hanßmann, Chapter 7).
To prove that $\Gamma^+(x)$ is bounded, we are going to write the system in polar coordinates. So, let $x = r\cos(\varphi), y = r\sin(\varphi)$. This implies that we have the following system:

$$\begin{cases} \dot{r}\cos(\varphi) - r\sin(\varphi)\dot{\varphi} = -r\cos(\varphi) + r^2\cos(\varphi)\sin(\varphi) \\ \dot{r}\sin(\varphi) + r\cos(\varphi)\dot{\varphi} = -r\sin(\varphi) - r^2\cos^2(\varphi) \end{cases}$$

Suppose we multiply the first row with $\tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)}$. This gives: $\dot{r}\sin(\varphi) - r\dot{\varphi}\frac{\sin^2(\varphi)}{\cos(\varphi)} = -r\sin(\varphi) + r^2\sin^2(\varphi)$.
Subtracting this from row 2: $\frac{r}{\cos(\varphi}\dot{\varphi} = -r^2$. Therefore, $\dot{\varphi} = -r\cos(\varphi)$.
Substituting this in the initial row 1: $\dot{r}\cos(\varphi) = -r\cos(\varphi)$, so $\dot{r} = -r$.
The system in polar coordinates is:

$$\begin{cases} \dot{r} = -r \\ \dot{\varphi} = -r\cos(\varphi) \end{cases}$$

Note that $\dot{r} < 0$ for all $r > 0$. So, each orbit starting with $r > 0$, will have a decreasing r. Also note that the only time $\dot{r} = 0$, is when $r = 0$. This implies that $x = 0, y = 0$, which is the equilibrium we found. So, consider the area $\{0 \leq r \leq R\}$ for some $R > 0 \in \mathbb{R}$. Note that since $\dot{r} < 0$ in this area , r is decreasing for all $r > 0$, and when $r = 0$, $\dot{r} = 0$, this area is forward invariant. When an orbit starts with $r > R > 0$, we still have that $\dot{r} < 0$, which implies that this orbit will also enter this area (and will never leave due to forward invariance). Therefore, $\Gamma^+(x)$ is bounded.

We now meet both requirements of the Poincaré-Bendixson Theorem which implies that the ω-limit set is one of the following possibilities:

- $\omega(l)$ is an equilibrium.
- $\omega(l)$ is a periodic orbit
- $\omega(l)$ consists of equilibria and orbits having these equilibria as their α- and ω-limit sets.

Note that the only equilibrium lies in $\{0 \leq r \leq R\}$. Note that in this area we have for all $r > 0, \dot{r} < 0$ and when $r = 0, \dot{r} = 0$ (which is at the only equilibrium (the stable double critical node)). Therefore, the Poincaré-Bendixson Theorem implies that $\omega(l) = \{(0, 0)\}$. □

3.2 Example 2

Suppose we have the following system in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = x(4 - x^2 - y^2) + y \\ \dot{y} = y(4 - x^2 - y^2) - x \end{cases}$$

Again, let $l = (x, y) \in \mathbb{R}^2$.
We can start with computing the equilibria of the system.
Note that to compute the equilibria of the system, we have the following:

$$\begin{cases} x(4 - x^2 - y^2) + y = 0 \\ y(4 - x^2 - y^2) - x = 0 \end{cases}$$

Suppose we multiply row 1 with y and row 2 with x. This gives:

$$\begin{cases} xy(4 - x^2 - y^2) + y^2 = 0 \\ xy(4 - x^2 - y^2) - x^2 = 0 \end{cases}$$

Now subtract row 2 of row 1 gives: $x^2 + y^2 = 0$. Note that this implies that $x = y = 0$. Therefore, the only equilibrium of this system in $\mathbb{R}^2$ is $(0, 0)$.
The Jacobian of the system is:

$$\begin{pmatrix} (4-x^2-y^2)-2x^2 & -2xy+1 \\ -2xy-1 & (4-x^2-y^2)-2y^2 \end{pmatrix}$$

At $(0,0)$ this is equal to

$$\begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}$$

To find the eigenvalues, we have to solve det $\begin{vmatrix} 4-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$. So, $(4-\lambda)^2 + 1 = 0$.
This implies that $\lambda^2 - 8\lambda + 16 + 1 = 0$. Using the abc-formula gives: $D = (-8)^2 - 4 \times 1 \times 17 = -4$.
So $\lambda = \frac{8 \pm 2i}{2}$. Therefore, $\lambda = 4 \pm i$. This implies that $(0,0)$ is an unstable focus (Kuznetsov and Hanßmann, Chapter 7).

To prove that $\Gamma^+(x)$ is bounded, we will first write the system in polar coordinates.

$$\begin{cases} \dot{r}\cos(\varphi) - r\sin(\varphi)\dot{\varphi} = r\cos(\varphi)(4-r^2) + r\sin(\varphi) \\ \dot{r}\sin(\varphi) + r\cos(\varphi)\dot{\varphi} = r\sin(\varphi)(4-r^2) - r\cos(\varphi) \end{cases}$$

When multiplying row 1 with $\tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)}$, we get: $\dot{r}\sin(\varphi) - r\dot{\varphi}\frac{\sin^2(\varphi)}{\cos(\varphi)} = r\sin(\varphi)(4-r^2) + r\frac{\sin^2(\varphi)}{\cos(\varphi)}$.
Subtracting this from row 2 gives: $\frac{r}{\cos(\varphi)}\dot{\varphi} = -\frac{r}{\cos(\varphi)}$ so $\dot{\varphi} = -1$.
Substituting this in the initial row 1 gives: $\dot{r}\cos(\varphi) + r\sin(\varphi) = r\cos(\varphi)(4-r^2) + r\sin(\varphi)$ so $\dot{r} = r(4-r^2)$.
The system in polar coordinates is:

$$\begin{cases} \dot{r} = r(4-r^2) \\ \dot{\varphi} = -1 \end{cases}$$

Note that $\dot{r} > 0$ if $r > 0$ and $4 > r^2$. This implies that $\dot{r} > 0$ if $0 < r < 2$.
Similarly, $\dot{r} < 0$ if $r(4-r^2) < 0$, so $r^2 > 4$. Therefore, $\dot{r} < 0$ if $r > 2$. Also, note that when $r = 2$, $\dot{r} = 0$.
Consider the area $\{0 < r < R\}$ with $R > 2$. Note that this area is forward invariant due to the fact that $\dot{r} < 0$ when $r > 2$, $\dot{r} = 0$ if $r = 2$ and $\dot{r} > 0$ when $0 < r < 2$. Therefore, if an orbit starts within this area, it will stay in this area. This implies that $\Gamma^+(l)$ is bounded in this area.

We can now apply the Poincaré-Bendixson theorem. This implies that the ω-limit set is one of the following possibilities:

- $\omega(l)$ is an equilibrium.
- $\omega(l)$ is a periodic orbit
- $\omega(l)$ consists of equilibria and orbits having these equilibria as their α- and ω-limit sets.

Note that the only equilibrium of the system $(0,0)$, does not lie in the forward invariant area $\{0 < r < R\}$. Therefore, option 1 and 3 are not possible. We can conclude that $\omega(l)$ is a periodic orbit.
However, we do not have to stop here. Note that $\dot{r} = 0$ if $r = 0$ or $r = 2$. When $r = 0$, we are at the equilibrium $(0,0)$. However, when $r = 2$, $\dot{r} = 0$ which implies that at $r = 2$ we are on the periodic orbit. So, $\omega(l)$ is a periodic orbit with $r = 2$. $\square$

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