The Poincaré-Bendixson Theorem

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1 Introduction

The Poincaré-Bendixson theorem is a classical result that describes the long term behaviour of bounded orbits in the plane. It is especially relevant as it rules out chaos for planar systems, limiting the evolution of bounded orbits to three cases and proving that there cannot exist strange attractors in $\mathbb{R}^2$.

A first version of the theorem, for polynomial systems, was proved by Henri Poincaré in [4] at the end of the XIX century. The proof was later completed by Ivar Bendixson in [1] for continuous systems. We prove the following version.

Theorem 1.1 (Poincaré-Bendixson)**.** *A bounded forward orbit of a smooth planar system*

$$\dot{X} = F(X) \ , \quad X \in \mathbb{R}^2 \ , \tag{1}$$

with a finite number of equilibria, as $t \longrightarrow +\infty$ tends to one of the following invariant sets in the phase plane:

1. *an equilibrium point;*
2. *a periodic orbit;*
3. *a union of equilibria and their connecting orbits.*

We start by defining ω-limit sets to describe the evolution of bounded orbits as $t \longrightarrow +\infty$ and prove relevant properties. We then define transverse lines and use geometric considerations, in particular the Jordan curve theorem, to prove a series of technical lemmas that constitute the backbone of the proof. By combining such results, we obtain an equivalent formulation of the theorem.

2 Preliminary results

2.1 ω-limit sets

We start by defining ω-limit sets and studying their properties.

Definition 2.1 (ω-limit sets)**.** Given a smooth system with flow φ^t, we say that $P \in \omega(X) \subset \mathbb{R}^2$ if there exists an increasing sequence of times (t_j), such that $t_j \longrightarrow +\infty$ and $P_j := \varphi^{t_j}(X) \longrightarrow P$ as $j \longrightarrow +\infty$. We call $\omega(X)$ the ω-limit set of the orbit starting at X.

We list here properties of ω-limit sets of a smooth systems hold in general for n-dimensional systems.

Property 2.2. Let $X \in \mathbb{R}^2$ and assume that the forward orbit of (1) starting at X is bounded, then

1. $\omega(X)$ is non-empty, bounded, closed, and connected;
2. $\omega(X)$ is an invariant set for the planar system, that is, if $Y \in \omega(X)$ then $\varphi^t(Y) \in \omega(X)$ for all $t \in \mathbb{R}$;
3. the forward orbit starting at X tends to $\omega(X)$, that is
$$\operatorname{dist}(\varphi^t(X), \omega(X)) = \inf_{P \in \omega(X)} ||\varphi^t(X) - P|| \longrightarrow 0 \ ,$$
as $t \longrightarrow +\infty$;
4. if $Z \in \omega(Y)$ and $Y \in \omega(X)$, then $Z \in \omega(X)$.

Proof. Let $X \in \mathbb{R}^2$ and assume that the forward orbit starting at X is bounded.

1. - As the orbit is bounded, there exists a bounded sequence $(\varphi^{t_j}(X))$ such that (t_j) is increasing and $t_j \longrightarrow +\infty$. Then, by the Bolzano-Weierstrass Theorem, $(\varphi^{t_j}(X))$ has a convergent subsequence $(\varphi^{t_{j_k}}(X))$. Let P be the limit of such convergence subsequence, then (t_{j_k}) is increasing, $t_{j_k} \longrightarrow +\infty$, and $(\varphi^{t_{j_k}}(X)) \longrightarrow P$, i.e. $P \in \omega(X)$. Therefore, $\omega(X) \neq \emptyset$.
   - By definition, $\omega(X)$ is a subset of the closure of the orbit. As the orbit is bounded, $\omega(X)$ is also bounded.
   - By definition, we can write the ω-limit set as
   $$\omega(X) = \bigcap_{\tau \geq 0} \overline{\{u(t) : t \geq \tau\}} \ ,$$
   where $u(t)$ is the solution to the system at time t. As it an intersection of closed sets, $\omega(X)$ is also closed.
   - We want to prove that $\omega(X)$ is connected, we do so by assuming it is not and finding a contradiction. Assume $\omega(X)$ is not connected, then, there exist open and disjoint A, B such that $\omega(X) \subset A \cup B$. Therefore, there exist increasing $(t_j), (s_j) \longrightarrow +\infty$ such that $t_j < s_j < t_{j+1}$ and $\varphi^{t_j}(X) \in A, \varphi^{s_j}(X) \in B$ for all j. Moreover,
   $$\{\varphi^t(X) : t \in [t_j, s_j]\}$$
   is a continuous curve connecting A and B. Since they are disjoint, for all j there exists $r_j \in]t_j, s_j[$ such that $\varphi^{r_j}(X) \notin A \cup B$. Then, $(r_j) \longrightarrow +\infty$ and it is increasing.
   As the orbit is bounded, the sequence $(\varphi^{r_j}(X))$ is also bounded, and by the Bolzano-Weierstrass Theorem it has a convergent subsequence $(\varphi^{r_{j_k}}(X)) \longrightarrow P$. As $X \backslash (A \cup B)$ is closed, $P \notin A \cup B$, but by definition $P \in \omega(X)$, this contradicts $\omega(X) \subset A \cup B$. Therefore, $\omega(X)$ is connected.
2. Assume $Y \in \omega(X)$. By definition, there exists an increasing sequence of times $(t_j) \longrightarrow +\infty$ such that $\varphi^{t_j}(X) \longrightarrow Y$ as $j \longrightarrow +\infty$. For all $t \in \mathbb{R}$, $(t + t_j)$ is also an increasing sequence of times that tends to $+\infty$ and, using the properties and continuity of the flow,
$$\varphi^{t+t_j}(X) = \varphi^t(\varphi^{t_j}(X)) \xrightarrow[j \to +\infty]{} \varphi^t(Y) \ .$$
Therefore, by definition $\varphi^t(Y) \in \omega(X)$ for all $t \in \mathbb{R}$.

3. By contradiction, suppose there exists $\varepsilon > 0$ and an increasing sequence $(t_j) \longrightarrow +\infty$ such that $\text{dist}(\varphi^{t_j}(X), \omega(X)) > \varepsilon$, but, as the orbit is bounded, $(\varphi^{t_j}(X))$ has a convergent subsequence $(\varphi^{t_{j_k}}(X)) \longrightarrow Q$ for some $Q \in \omega(X)$. This contradicts $\text{dist}(\varphi^{t_j}(X), \omega(X)) > \varepsilon$, therefore $\text{dist}(\varphi^{t}(X), \omega(X)) \longrightarrow 0$.

4. As $Y \in \omega(X)$, by invariance $\varphi^t(Y) \in \omega(X)$ for all $t \in \mathbb{R}$. As $Z \in \omega(Y)$, there exists an increasing sequence $(t_j) \longrightarrow +\infty$ such that $\varphi^{t_j}(Y) \longrightarrow Z$, but $\varphi^{t_j}(Y) \in \omega(X)$ for all t_j and $\omega(X)$ is closed, therefore $Z \in \omega(X)$.

□

2.2 Transverse lines and the Jordan curve theorem

In this section we define transverse lines and use geometric considerations, in particular the Jordan curve theorem, to prove three technical lemmas required to complete the proof.

Definition 2.3 (Transverse line)**.** A closed segment L is called transverse for (1) if the vector field F does not vanish on L and it is nowhere tangent to L.

Definition 2.4 (Jordan curve)**.** A Jordan curve is the image of a continuous map $\varphi : [0,1] \to \mathbb{R}^2$ such that $\varphi(0) = \varphi(1)$ and the restriction of φ to $[0, 1[$ is injective. That is, a planar closed curve without self-intersections.

The following theorem can be found in [5] and we report it here without proof, as the proof is quite long and not particularly relevant in this context.

Theorem 2.5 (Jordan Curve Theorem)**.** *Let Γ be a Jordan curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \backslash \Gamma$ is disconnected and consists of exactly two connected components, the interior (bounded) and the exterior (unbounded).*

The following lemma is the main result we need to prove the Poincaré-Bendixson Theorem, in particular it requires the Jordan Curve Theorem, that holds only in $\mathbb{R}^2$. This is the main reason why the theorem does not generalise for higher dimensions.

Lemma 2.6 (Monotonicity)**.** *If an orbit intersects L for an increasing sequence of times (t_j), then the corresponding sequence of intersection points (Q_j) is either constant or strictly monotone.*

Proof. We partially follow a proof from [5]. If $Q_0 = Q_1$, then the orbit is periodic and (Q_n) is constant.

Assume $Q_0 \neq Q_1$ and define the curve Γ as the union of the the section of the orbit from Q_0 to Q_1 along $\varphi^t(Q_0)$ and the segment from Q_1 to Q_0 along the transverse L. Then, it is a closed curve, without self intersections because of uniqueness of solutions and the definition of transverse line, i.e. it is a Jordan curve. By the Jordan Curve Theorem 2.5, Γ divides $\mathbb{R}^2$ into two connected components D_1 and D_2.

Let $u(t)$ be the solution to the system (1) starting from Q_0. Since $F(Q_1)$ is transversal to L, the vector field F points either to D_1 or D_2 and u must enter either D_1 or D_2 after t_1. Call D_1 the region that u enters (either the interior or the exterior), we claim that $u(t) \in D_1$ for all $t > t_1$. By contradiction, assume there exists $t_2 > t_1$ such that $u(t_2) \in D_2$, therefore there is a $t^* \in]t_1, t_2[$ such that $u(t^*) \in \Gamma$ by the Intermediate Value Theorem. But this is impossible, because $u(t^*)$ cannot belong to the orbit as it would contradict uniqueness of solutions and it cannot belong to the segment as F points towards D_1 on L.

Therefore, $Q_2 = u(t_2) \in D_1$ and Q_0, Q_1, Q_2 are strictly monotone along L. By iterating this procedure, we prove that the sequence (Q_j) is strictly monotone.

□

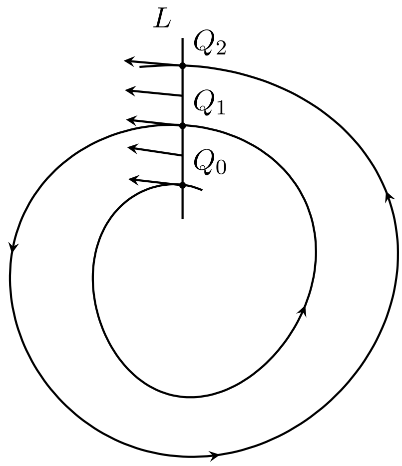

Figure 1: Lemma 2.6.

Lemma 2.7. *Let $P \in \omega(X)$ and let L be a transverse segment through P. Then there is an increasing sequence $(t_j) \longrightarrow +\infty$ such that, for $Q_j = \varphi^{t_j}(X)$, we have $(Q_j) \longrightarrow P$ and $Q_j \in L$ for all j.*

Proof. First, we prove a result that holds for all $P \in L$, and later apply it in particular to $P \in \omega(X) \cap L$. Let $P \in L$ and let (y, z) be the coordinates on $\mathbb{R}^2$. Without loss of generality, assume that $P = (0, 0)$ and L is a subset of $\{(y, z) : y = 0\}$. For ε sufficiently small to be determined, define a map $\psi : B_{\mathbb{R}}(0, \varepsilon) \times B_{\mathbb{R}^2}(0, \varepsilon) \to \mathbb{R}$ by

$$\psi(t, X) = \pi(\varphi^t(X)) \ ,$$

where $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the projection such that $\pi(y, z) = y$. Then, ψ is a $\mathcal{C}^1$ map and, by definition, $\psi(0, 0) = 0$. Moreover, as L is transversal, $\frac{\partial \psi}{\partial t}(0, 0) = \pi(F(X)) \neq 0$. Therefore, by the implicit function theorem, there exists an open set U with $P \in U$ and a $\mathcal{C}^1$ function $\tau : U \to \mathbb{R}$ such that $\varphi^{\tau(X)}(X) \in L$ for all $X \in U$.

Now consider initial conditions X and let $P \in \omega(X)$, L transverse line through P and U the open neighbourhood of P found from the previous part of the proof. Then, by definition of ω-limit set, there exists an increasing sequence of times $(t_j) \longrightarrow +\infty$ such that $\varphi^{t_j}(X) \longrightarrow P$ as $t_j \to +\infty$ and $\varphi^{t_j}(X) \in U$ for all t_j. Therefore

$$Q_j := \varphi^{\tau(\varphi^{t_j}(X))}(\varphi^{t_j}(X)) = \varphi^{\tau(\varphi^{t_j}(X)) + t_j}(X) \in L$$

for all j and $|\tau(\varphi^{t_j}(X))| < \varepsilon$.

Following the same argument used to prove Lemma 2.6, we find that the sequence of times $\tau(\varphi^{t_j}(X)) + t_j \longrightarrow +\infty$ is increasing, possibly with repeated elements that we can remove and redefine (Q_j) accordingly. Then, by Lemma 2.6, the sequence $(Q_j) \subset L$ is monotone and, as L is closed, (Q_j) converges to P, proving the claim.

□

Lemma 2.8. *The set $\omega(X)$ can intersect a transverse segment L in at most one point.*

Proof. Assume there are two points of intersection P_1 and P_2. Then, there exist sequences $(Q_{1,j})$,$(Q_{2,j})$ in the intersection of L and the orbit, converging to P_1 and P_2 respectively. But this is impossible since both are subsequences of a monotone sequence (Q_j) from Lemma 2.6, and a monotone sequence cannot have two accumulation points.

□

3 The Poincaré-Bendixson Theorem

We now use the previous lemmas to prove two theorems that cover the last two cases of the Poincaré-Bendixson Theorem, namely the case when $\omega(X)$ is a periodic orbit and the case when it is a union of equilibria and their connecting orbits.

Theorem 3.1 (Periodic orbits)**.** *If $\omega(X)$ does not contain equilibria, then it contains a periodic orbit Γ_0. Moreover, $\omega(X) = \Gamma_0$.*

Proof. We follow the proof from [2]. Let $Y \in \omega(X)$ and consider $Z \in \omega(Y)$ (they exist because of Property 2.2.1), then, by Property 2.2.2, $Z \in \omega(X)$, therefore it is not an equilibrium. Take L a transverse through Z and a sequence

$$Y_j = \varphi^{t_j}(Y) \longrightarrow Z$$

such that $Y_j \in L$ for all t_j, that exists thanks to Lemma 2.7. But as $Y_j \in L$, by Lemma 2.6 the sequence (Y_j) is either monotone or constant. It cannot be monotone as this would mean that $\omega(X)$ and L have more than one intersection, contradicting Lemma 2.8, therefore (Y_j) is constant and equal to Z. This implies that the orbit starting at Y is a periodic orbit and we denote it by Γ_0.

There is a transverse segment M_Y through any point $Y \in \Gamma_0$ and, by Lemma 2.8, $\omega(X) \cap M_Y = Y$. As for every point in Γ_0 there is a closed transverse segment passing through it, it is possible to trap Γ_0 in an open annulus $A \subset \bigcup_{Y \in \Gamma_0} M_Y$, and by choosing the length of the segments to be sufficiently small, an open annulus in which Γ_0 is the only subset of $\omega(X)$. Since $\omega(X)$ is connected (Property 2.2.1), $\omega(X) \backslash \Gamma_0$ is empty, proving the claim.

□

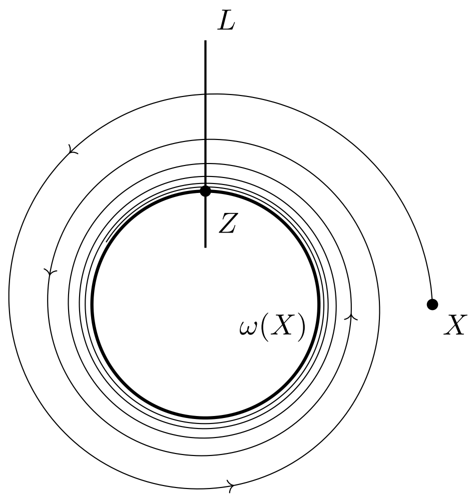

Figure 2: Theorem 3.1.

We now define α-limit sets, an analogous to ω-limit sets for negative and decreasing times, that we will need to prove the final part of the theorem.

Definition 3.2 (α-limit sets)**.** Given a smooth system with flow φ^t, we say that $P \in \alpha(X) \subset \mathbb{R}^2$ if there exists a decreasing sequence of negative times $\{t_j\}$, such that $t_j \longrightarrow -\infty$ and $P_j := \varphi^{t_j}(X) \longrightarrow P$ as $j \longrightarrow +\infty$. We call $\alpha(X)$ the α-limit set of the orbit starting at X.

Analogous properties and results to ω-limit sets hold. Consider the system with reversed time, then it has the same orbits but followed in reversed time. In particular, every increasing sequence of times (t_j) for the reversed system, such that $t_j \longrightarrow +\infty$ and $P_j := \varphi^{t_j}(X) \longrightarrow P$ as $j \longrightarrow +\infty$ corresponds to a decreasing sequence of times (t_j) for the original system, such that $t_j \longrightarrow -\infty$ and $P_j := \varphi^{t_j}(X) \longrightarrow P$ as $j \longrightarrow -\infty$.

Then, for all $X \in \mathbb{R}^2$, $\alpha(X)$ in the original system is equal to $\omega(X)$ in the time-reversed system. Therefore the same properties of ω-limit sets hold for α-limit sets (with reversed time).

Theorem 3.3 (Union of equilibria and their connecting orbits)**.** *If $\omega(X)$ contains both equilibrium and non-equilibrium points and $Q \in \omega(X)$ is not an equilibrium, then both $\omega(Q)$ and $\alpha(Q)$ are equilibrium points.*

Proof. We follow the proof from [2]. By Property 2.2.4, as $Q \in \omega(X)$, both $\omega(Q) \subset \omega(X)$ and $\alpha(Q) \subset \omega(X)$. By contradiction, if $\omega(Q)$ does not contain equilibria, then it is a periodic orbit, and thus $\omega(X)$ is also a periodic orbit, because of Theorem 3.1. But we assumed that $\omega(X)$ contains equilibria, a contradiction. Therefore, $\omega(Q)$ contains an equilibrium.

Assume that $\omega(Q)$ also contains a non equilibrium point Y and take a transversal L through Y. Because of Property 2.2.2, all point of the orbit starting at Q belong to $\omega(X)$. By Lemma 2.7, the forward orbit starting at Q intersects L infinitely many times in a neighbourhood of Y, and these intersection points must be different as otherwise the orbit would be a periodic orbit, but we have proved that it is not. This means that $\omega(X)$ intersects L more than once, a contradiction.

Therefore, $\omega(Q)$ contains only equilibria, thus, it is a union of a finite number of points. However, $\omega(Q)$ is connected thanks to Property 2.2.1, so it is just one equilibrium. A completely analogous argument proves that $\alpha(Q)$ is also a single equilibrium point. □

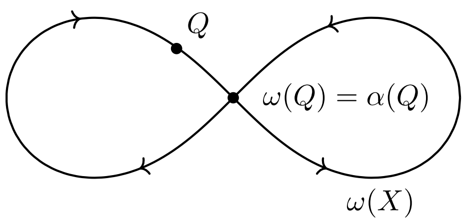

Figure 3: Theorem 3.3, particular case where $\omega(Q) = \alpha(Q)$.

Finally, combining the previous results we obtain the following version of the Poincaré-Bendixson Theorem, that is equivalent to the one formulated in the introduction thanks to properties 2.2.2 and 2.2.3.

Theorem 3.4 (Poincaré-Bendixson)**.** *Consider a smooth planar system that has only a finite number of equilibrium points. Suppose that a forward orbit of this system starting at a point $X \in \mathbb{R}^2$ is bounded. Then $\omega(X)$ is either*

1. *an equilibrium point;*
2. *a periodic orbit;*
3. *a union of equilibria and orbits having these equilibria as their α- and ω-limit sets.*

Proof. We want to verify that the first case is possible. Let P be an equilibrium point, then $\varphi^t(P) = P$ for all $t > 0$, therefore $\omega(P) = P$ is a single equilibrium point. Moreover, if $\omega(X)$ contains only equilibria, it contains exactly one as it is connected by Property 2.2.1.

If $\omega(X)$ does not contain only equilibria (i.e. is not a single equilibrium point), then it either does not contain equilibria or it contains both equilibrium and non-equilibrium points.

If $\omega(X)$ does not contain equilibria, then it is a periodic orbit by Theorem 3.1. If $\omega(X)$ contains both equilibrium and non-equilibrium points then, by Theorem 3.3, if $Q \in \omega(X)$ is not an equilibrium both $\omega(Q)$ and $\alpha(Q)$ are equilibrium points. This means that Q belongs to an orbit connecting the equilibria $\omega(Q)$ and $\alpha(Q)$, therefore $\omega(X)$ is a union of equilibria and their connecting orbits.

We have covered all the possible cases, proving the theorem.

□

4 Examples and behaviour on other manifolds

4.1 Examples

For the first case, consider the system

$$\begin{cases} \dot{x} = x(6 - x^2 - y^2) - 3y^2 \\ \dot{y} = y(4 - x^2 - y^2) + 2x \end{cases},$$

it has three equilibria: a stable node in $(-2.5, -1.3)$, an unstable node in $(0, 0)$ and a saddle in $(1, -1.1)$. As shown in Figure (4), bounded forward and backwards orbits tend to one and only one of the equilibria.

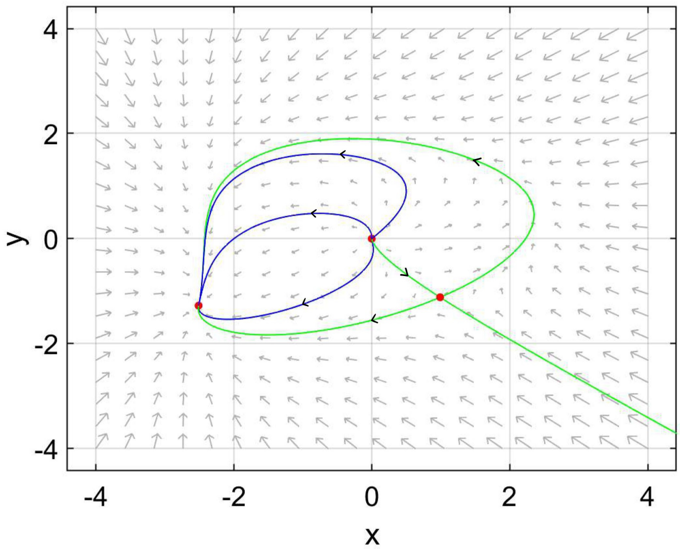

Figure 4: Phase portrait showing equilibria that are ω- and α-limit sets for different initial conditions.

Now, for the second case, consider the slightly different system

$$\begin{cases} \dot{x} = x(6 - x^2 - y^2) - 3y^3 \\ \dot{y} = y(4 - x^2 - y^2) + 2x \end{cases},$$

it has one unstable equilibrium in $(0,0)$ and, as we observe in Figure (5), bounded forward orbits tend to a periodic orbit.

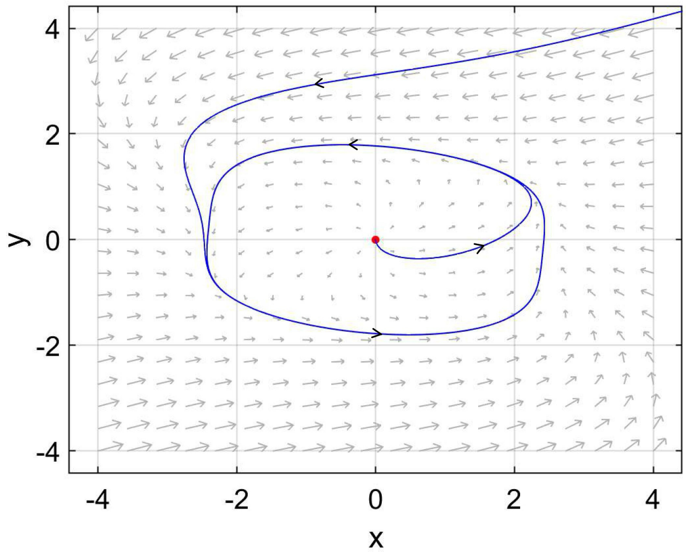

Figure 5: Phase portrait showing a periodic orbit that is ω-limit set for different forward orbits.

Finally, for the third case consider the system

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} + \mu H \frac{\partial H}{\partial x} \\ \dot{y} = -\frac{\partial H}{\partial x} + \mu H \frac{\partial H}{\partial y} \end{cases},$$

with $H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$ and $\mu = -0.1$. It has a saddle point in $(0,0)$ and bounded forward orbits tend the saddle and two homoclinic orbits connecting the saddle to itself, as we observe in Figure (6).

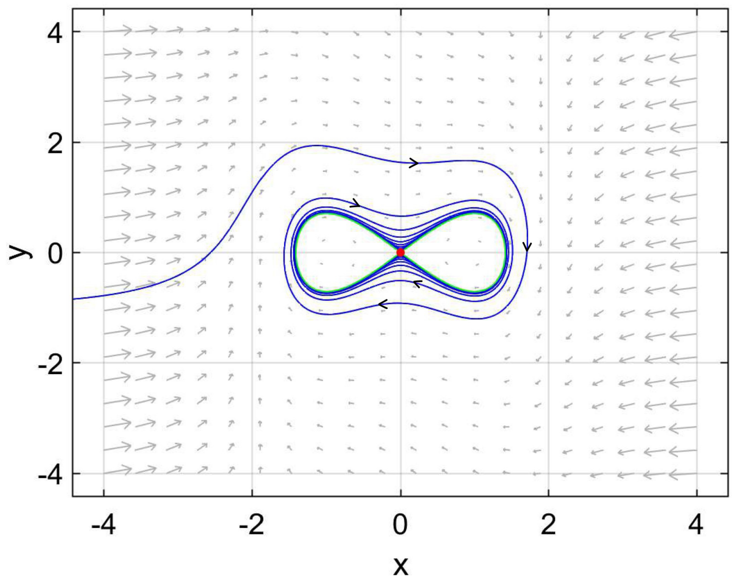

Figure 6: Phase portrait showing a forward orbit tending to the union of a saddle equilibrium and two homoclinic orbits.

4.2 On different manifolds

We have proved that the Poincaré-Bendixson Theorem holds for differential systems in $\mathbb{R}^2$, it is now natural to ask whether similar results hold for other manifolds. We show some examples without going into detail about the proofs.

The theorem does not hold in higher dimensions, for example in $\mathbb{R}^3$, and famous examples are strange attractors and the Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases},$$

that shows chaotic behaviour for $\rho = 28$, $\sigma = 10$, $\beta = \frac{8}{3}$.

Regarding two-dimensional manifolds, an analogue of the theorem holds for the sphere and the cylinder, but not for the torus, as shown in [3]. For example, on the torus the smooth system in toroidal-poloidal coordinates

$$\begin{cases} \dot{\theta}_1 = 1 \\ \dot{\theta}_2 = \nu \end{cases},$$

when ν is irrational produces dense orbits.

References

[1] Ivar Bendixson. "Sur les courbes définies par des équations différentielles". In: *Acta Mathematica* 24 (1901), pp. 1–88. DOI: `10.1007/BF02403068`. URL: `https://doi.org/10.1007/BF02403068`.

[2] Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*. Springer Cham, 2023, pp. 37–41.

[3] James D. Meiss. *Differential Dynamical Systems*. Society for Industrial and Applied Mathematics, 2007.

[4] Henri Poincaré. "Sur les courbes définies par les équations différentielles". fr. In: *Journal de Mathématiques Pures et Appliquées* 4e série, 1 (1885), pp. 167–244. URL: `http://www.numdam.org/item/JMPA_1885_4_1__167_0/`.

[5] Gerald Teschl. *Ordinary Differential Equations and Dynamical Systems*. American Mathematical Society, 2011, pp. 220–223.