Bifurcations of Limit Cycles

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Bifurcations of Limit Cycles (1)

Definition 1: A Floquet multiplier m is called

- *hyperbolic* if $|m| \neq 1$;
- *elliptic* if $|m| = 1$ but $m \notin \{-1, 1\}$;
- *parabolic* if $m \in \{1, -1\}$.

Fact 1: where the number of **hyperbolic multipliers changes**, the family of periodic orbits undergoes a **bifurcation**. In a generic one-parameter family, this may happen in **three ways**:

1. a pair of Floquet multipliers on the unit circle meets at $+1$ and passes to the positive real axis;
2. a pair of Floquet multipliers on the unit circle meets at -1 and passes to the negative real axis;
3. two pairs of Floquet multipliers meet in a *Krein collision* at a double pair and turn into a quartet $(\gamma, \overline{\gamma}, \gamma^{-1}, \overline{\gamma}^{-1})$.

Bifurcations of limit cycles (2)

Consider

$$H(x,y,q,p) = \omega y + \frac{a}{2}p^2 + \frac{b}{6}q^3 + dyq,$$

where $\omega > 0$ and $a, b, d \neq 0$.
Reducing the S^1-symmetry:

$$\mathcal{H}_y(q,p) = \frac{a}{2}p^2 + \frac{b}{6}q^3 + dyq.$$

Remark 1: the same qualitative description remains true for the more general Hamiltonian

$$N(x,y,q,p) = \eta(y) + \mathcal{N}_y(q,p)$$

$$\mathcal{N}_y(q,p) = \frac{a(y)}{2}p^2 + \frac{b(y)}{6}q^3 + c(y)q + \text{h.o.t.}$$

where $a(0), b(0) \neq 0$, $c(0) = 0$, $d(0) := c'(0) \neq 0$.

Bifurcations of limit cycles (3)

Theorem 3.1.
Let H be a Hamiltonian system in two degrees of freedom and let γ be a periodic orbit of X_H with all Floquet multipliers $+1$.
Under generic conditions on H, a **periodic centre-saddle bifurcation** takes place as the value of H passes through $H(\gamma)$.

Proof: Use Floquet coordinates around $\gamma = \{(y, q, p) = (0, 0, 0)\}$.Note: $\dot{x} = \partial_y H(x, 0, 0, 0)$ along γ. This determines the *speed* at which γ is traversed, but not γ itself.

$$\omega := \int_0^1 \partial_y H(x, 0, 0, 0) dx \overset{!}{\neq} 0.$$

Then γ is also described by $\dot{x} = \omega$.

$$H = N + P,$$

where N is as before and P is at least fourth order in q and p.
Under genericity conditions $a, b, d \neq 0$, N undergoes periodic centre-saddle bifurcation.
One can show that this bifurcation can be lifted to H. ■

Bifurcations of limit cycles (4)

Consider the *Energy-Momentum mapping*

$$\mathcal{EM} : S^1 \times \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, q, p) \mapsto (H(x, y, q, p), y).$$

The singular values of $\mathcal{EM}$ form the following curves.

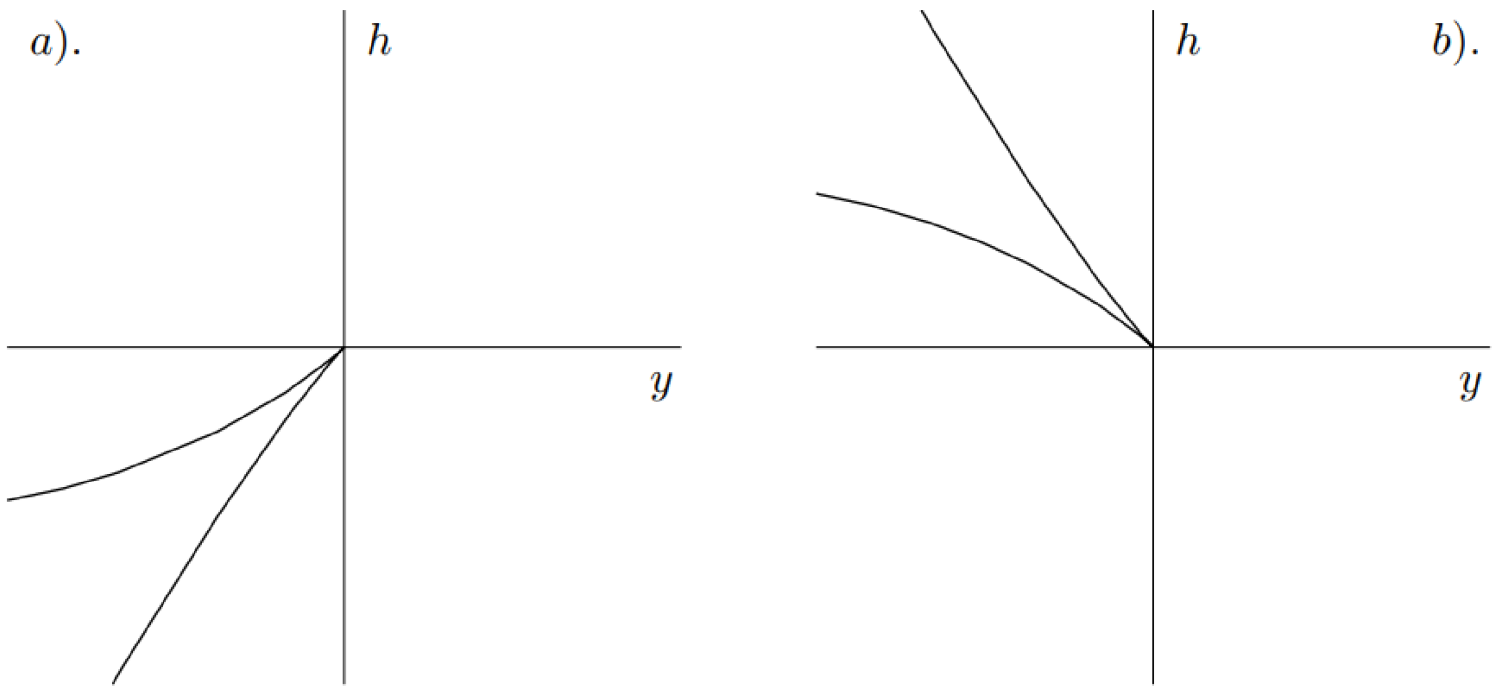

Fig. 3.1. Singular values of the energy-momentum mapping $\mathcal{EM} = (H, y)$ of (3.1) for positive a, bd and (**a**) positive ω, (**b**) negative ω.

Bifurcations of limit cycles (5)

Consider the Hamiltonian

$$H(x,y,q,p) = \omega y + \frac{1}{2}p^2 + \frac{1}{24}q^4 + \frac{1}{2}yq^2, \quad (x,y,q,p) \in S^1 \times \mathbb{R}^3.$$

Reducing the S^1-symmetry:

$$\mathcal{H}_y(q,p) = \frac{1}{2}p^2 + \frac{1}{24}q^4 + \frac{1}{2}yq^2, \quad (q,p) \in \mathbb{R}^2.$$

Reducing the $\mathbb{Z}_2$-symmetry:

$$\mathcal{H}_y(u,v,w) = v + \frac{1}{6}u^2 + yu, \quad (u,v,w) \in \mathbb{R}^2/\mathbb{Z}_2.$$

$$\begin{array}{ccc} S^1 \times \mathbb{R}^3 & \xrightarrow{(x,y,q,p)\sim(x+\frac{1}{2},y,-q,-p)} & S^1 \times \mathbb{R} \times \mathbb{R}^2/\mathbb{Z}_2 \\ \Big\downarrow S^1\text{-symmetry} & & \Big\downarrow S^1\text{-symmetry} \\ \{y\} \times \mathbb{R}^2 & \xrightarrow{(q,p)\sim(-q,-p)} & \{y\} \times \mathbb{R}^2/\mathbb{Z}_2 \end{array}$$

Bifurcations of limit cycles (6)

Theorem 3.4.
Let H be a Hamiltonian system in two degrees of freedom and let γ be a periodic orbit of X_H with Floquet multipliers -1.
Under generic conditions on H, a **period-doubling bifurcation** takes place as the value of H passes through $H(\gamma)$.

Proof: Similar to Theorem 3.1, see lecture notes.

Bifurcations of limit cycles (7)

Consider the *Energy-Momentum mapping*

$$\mathcal{EM} : S^1 \times \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, q, p) \mapsto (H(x, y, q, p), y).$$

The singular values of $\mathcal{EM}$ form the following curves.

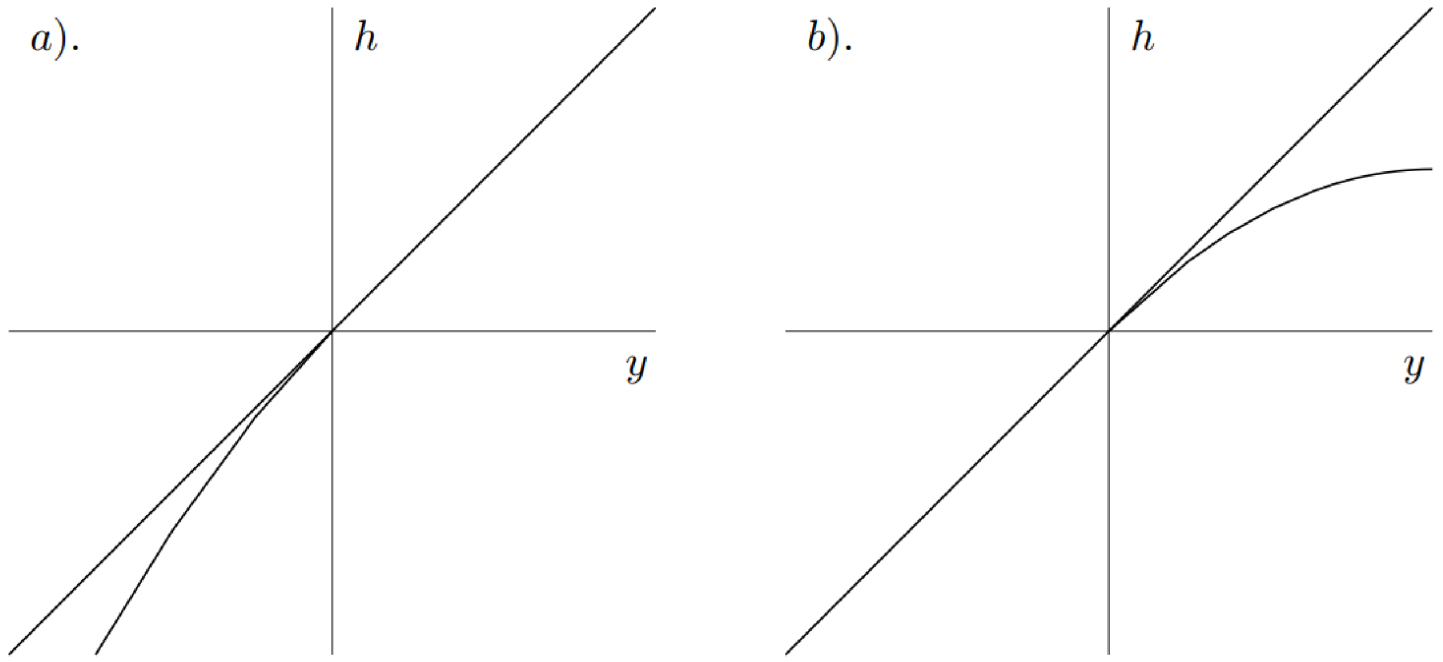

Fig. 3.2. Singular values of the energy-momentum mapping $\mathcal{EM} = (H, y)$ of (3.3) for positive ω, a, b and (**a**) positive d, (**b**) negative d.