INLDS: Final Assignment (Revised)

Period-3 implies chaos

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1 Introduction

Suppose that you are a population biologist, unhappy with the state of existing models on population growth. Most of them either assume exponential growth or the collapse of the population in the long run, which makes you a bit uneasy. In order to remedy this, you construct a more "natural" model, in which the population P_t at time t will tend to grow if it it below some limiting value L, and decrease if it exceeds it (e.g., due to overcrowding, limited resources, etc). To better control the model, you also add a parameter that determines the rate of change, call it k, and you also normalize $L = 1$, since units are arbitrary anyways. What you arrive at is excessively simple: $P_{t+1} = kP_t(1 - P_t)$ [1]. When you run a bunch of simulations, however, you find out that you get a whole lot more than you bargained for.

Models of similar form, only change the letters, arise naturally in a plethora of disciplines and fields of study, from epidemiology through finance to neural networks. More specifically, what we have here is a discrete-time dynamical system generated by the iterated application of what is called the logistic map. And while this map appears very simple, a closer look reveals that it exhibits significant regime switching, and small changes in the parameter k can significantly affect its predictions.

You run a simulation for some relatively small k, and you find that the model predicts a collapse of the population. You run another one for a higher k, and suddenly the model predicts convergence to some positive equilibrium size of the population. You increase k a bit more, and you find a cycle, along which the population size will increase and decrease regularly. Then, things start to get confusing: the cycle that your simulation converges to becomes more and more complicated, until the dynamics descends into chaos, only interrupted by seemingly unpredictable periods of stability.

A look at this model raises some important questions. For example, a more advanced simulation would tell you that, at many parameter values, at which we easily observe a stable cycle, there are also many more unstable cycles. Are there ever infinitely many of them, and can we ever say this with confidence? One result that addresses this question, first introduced by Li & Yorke, seems like magic due to its minimal assumptions. It states that, as long as a continuous mapping has a cycle of minimal period 3, it has a cycle of minimal period n for all $n \geq 1$.

The goal of the present paper is to demystify this result, apply it to the logistic map, by showing the existence of a fold bifurcation which generates a period-3 cycle, and then, to a lesser extent, to provide an overview of the behavior of this system more generally, so that the fold bifurcation and Li & Yorke's result can be better contextualized. But first, it is going to introduce some necessary concepts related to discrete-time dynamical systems.

2 Discrete dynamical systems

A dynamical system can be broadly defined as a system, whose state evolves in time according to a predefined, fixed rule. This rule can be given by a (possibly multi-dimensional) differential equation, which treats time as a continuous variable, or by a function, whose output is the state of the system one unit of "time" later. Systems generated by the second type are commonly called "discrete-time dynamical systems," and are the object of study of the present project. Before proceeding with the main text, it would be necessary to define some concepts related to this class of systems [2], assuming that the reader has knowledge of continuous-time dynamical systems, but not discrete dynamical systems. For the sake of maintaining the focus, a lot of unnecessary detail is omitted, and only one-dimensional maps are discussed.

Consider the function $f : I \to I$, $x \mapsto f(x)$, $I \subseteq \mathbb{R}$ closed. Let us write $f^0(x) = x$, $f^1(x) = f(x)$, then the composition of f with itself n times is denoted $f^n(x) = f(f^{n-1}(x))$, and we call it the n'th iterate of f. Given an initial condition x_0, we define the forward orbit of x_0 by f as the set of all iterates of x_0, and denote it as $O_f^+(x_0) = \{f^j(x_0) : j \geq 0\}$.

A point $p \in I$ is called a fixed point (or a period-1 point) of f if $f(p) = p$. Analogously, a point $p \in I$ is called a period-n point of f if $f^n(p) = p$, but $f^j(p) \neq p$ for $0 < j < n$. If p is a period-n point, then $O_f^+(p) = \{f^j(p) : 0 \leq j < n\}$ is called a period-n orbit (or cycle), and n is called the (minimal) period of the orbit. Notice that if p is a period-n point, all points in its orbit are also period-n points. Furthermore, period-n points are by definition fixed points of f^n, but it is possible that f^n also has other fixed points, corresponding to orbits of smaller periods. A given period-n point necessarily gives rise to a period-n cycle.

For discrete-time dynamical systems, notions of stability are defined similarly to the continuous-time case. An important property is the following: let f be continuously differentiable, and p_0 be a period-n point of f, for $n \geq 1$ (and denote $p_j = f^j(p_0)$ for $1 \leq j < n$, if $n > 1$). If $|(f^n)'(p_0)| < 1$, then the period-n point p_0 is stable; if $|(f^n)'(p_0)| > 1$, it is unstable. Notice that by the chain rule $|(f^n)'(p_0)| = |f'(p_{n-1})|...|f'(p_1)||f'(p_0)|$, and thus equal for all points along the periodic orbit of p_0; this means that we can talk about the whole periodic orbit being stable or unstable itself, depending on the stability of any period-n point along it.

The notion of a bifurcation is also defined similarly to the continuous-time case, and represents a qualitative change in the behavior of the system as its parameters are varied. We encounter three types of bifurcations in this paper, so let us briefly discuss them and the conditions, under which they arise, to the extent that would be useful for the analysis.

The first type of bifurcation is the fold bifurcation (also called tangential, or saddle-node), under which two fixed points or periodic orbits, one stable and one unstable, are created (or collide and disappear). Assume that $f(x, \mu)$ is a C^2 function from $I \times \mathbb{R}$ to I, such that $f(x_0, \mu_0) = x_0$, i.e. x_0 is a fixed point for the given parameter value μ_0, and furthermore, that the derivatives of f satisfy the following: $f_x(x_0, \mu_0) = 1$; $f_{xx}(x_0, \mu_0) \neq 0$; $f_\mu(x_0, \mu_0) \neq 0$. Then a fold bifurcation occurs at μ_0; furthermore, if $-f_{xx}(x_0, \mu_0)/f_\mu(x_0, \mu_0) > 0$, the fixed points appear for $\mu > \mu_0$; if $-f_{xx}(x_0, \mu_0)/f_\mu(x_0, \mu_0) < 0$, they appear for $\mu < \mu_0$. If we consider iterates f^n, a fold bifurcation concerning period-n points would correspond to the creation (or collision of disappearance) of period-n points and thus cycles. Such a fold bifurcation of cycles arises under the same conditions as above, applied, of course, to f^n, requiring that x_0 is a period-n point.

The second type of bifurcation is the period doubling bifurcation, under which a period-n orbit (or fixed point, if $n = 1$) loses stability, and a period-$2n$ orbit is created. Assume that $f(x, \mu)$ is a C^3 function from $I \times \mathbb{R}$ to I, such that $f(x_0, \mu_0) = x_0$, i.e. x_0 is a fixed point for the given parameter value μ_0, and furthermore, that the derivatives of f satisfy the following: $f_x(x_0, \mu_0) = -1$; $\alpha = [f_{\mu x} + \frac{1}{2}(f_\mu)(f_{xx})]_{(x_0,\mu_0)} \neq 0$; $\beta = (\frac{1}{3!} f_{xxx}(x_0, \mu_0)) + (\frac{1}{2!} f_{xx}(f_0, \mu_0))^2 \neq 0$. Then a period-doubling bifurcation occurs at μ_0; furthermore, if $-\beta/\alpha > 0$, the period-2 orbit appears for $\mu > \mu_0$; if $-\beta/\alpha < 0$, it appears for $\mu < \mu_0$. Once again, if we consider iterates f^n, a period-doubling bifurcation would imply the loss of stability of the given period-n point and thus cycle, and the creation of a period-$2n$ cycle; the conditions are again the same, but applied on the iterate f^n.

Finally, we can introduce the notion of a transcritical bifurcation, under which two fixed points collide and exchange stability. Once again, assume that $f(x, \mu)$ is a C^2 function from $I \times \mathbb{R}$ to I, such that $f(x_0, \mu_0) = x_0$. Furthermore, assume that $f_x(x_0, \mu_0) = 1$; $f_{xx}(x_0, \mu_0) \neq 0$; $f_{x\mu}(x_0, \mu_0) \neq 0$. Then a transcritical bifurcation occurs at μ_0, described qualitatively as above.

3 Period-3 implies chaos

In this section, we prove the following theorem [3]:

Theorem 1 (Li & Yorke)**.** *Suppose a continuous mapping $f : [0, 1] \to [0, 1]$ has a cycle of minimal period 3. Then f has a cycle of minimal period n for all $n \geq 1$.*

To do so, let us consider the continuous mapping $f : I \to I$, $I = [0, 1]$, as per the theorem, and let $J, K \subset I$ be two closed intervals. First, we introduce the following definition.

Definition 1. *We say that J **covers** K **under** f and write $J \rightharpoonup K$, if there exists a closed interval $L \subset J$ such that $f(L) = K$.*

This yields the following intermediate results.

Lemma 1. *If $J \rightharpoonup J$ under f, then f has a fixed point $x \in J$.*

Proof. Let $J = [a, b]$. Then, by definition, there exists a closed interval $L \subset J$, such that $f(L) = J$. This implies that there exist $c, d \in L$, such that $f(c) = a \leq c$ and $f(d) = b \geq d$.

Let us define the function $g(x) = f(x) - x$; it is the difference of two continuous functions, and thus continuous. Furthermore, by construction, $g(c) = f(c) - c \leq 0$, and $g(d) = f(d) - d \geq 0$. Thus, by the intermediate value theorem, there must be an $x \in L$ such that $g(x) = 0$, i.e. $f(x) = x$. ∎

Lemma 2. *If $I_0 \rightharpoonup I_1 \rightharpoonup I_2 \rightharpoonup \ldots \rightharpoonup I_n$ under f, then there exists a closed interval $J \subset I_0$ such that $f^k(J) \subset I_k$ for $k = 1, 2, ..., n-1$, and $f^n(J) = I_n$.*

Proof. By definition of $I_0 \rightharpoonup I_1$, there exists a closed interval $J_0^1 \subset I_0$ such that $f(J_0^1) = I_1$. So the claim is satisfied for $n = 1$.

Then, by definition of $I_1 \rightharpoonup I_2$, there exists a closed interval $J_1^2 \subset I_1$ such that $f(J_1^2) = I_2$. But $J_1^2 \subset I_1 = f(J_0^1)$, so there must exist a closed subset $J_0^2 \subset J_0^1 \subset I_0$ such that $f(J_0^2) = J_1^2$. Thus $f^2(J_0^2) = f(J_1^2) = I_2$, and furthermore, $f(J_0^2) = J_1^2 \subset I_1$, so the claim is also satisfied for $n = 2$.

We proceed inductively. Assume that for some $n = j \geq 2$ there exists a $J_0^j \subset I_0$ such that $f^k(J_0^j) \subset I_k$ for all $k = 1, 2, ..., j-1$, and $f^j(J_0^j) = I_j$. Now consider $j+1$. Since $I_j \rightharpoonup I_{j+1}$, there exists a closed interval $J_j^{j+1} \subset I_j$ such that $f(J_j^{j+1}) = I_{j+1}$. But $J_j^{j+1} \subset I_j = f^j(J_0^j)$, so there must exist $J_0^{j+1} \subset J_0^j$ such that $f^j(J_0^{j+1}) = J_j^{j+1}$. Then $f^{j+1}(J_0^{j+1}) = f(f^j(J_0^{j+1})) = f(J_j^{j+1}) = I_{j+1}$. Furthermore, $J_0^{j+1} \subset J_0^j$, so $f^k(J_0^{j+1}) \subset I_k$ for all $k = 1, 2, ..., j$.

Thus, for any $n \geq 1$, we can find J_0^n such that $f^n(J_0^n) = I_n$, and $f^k(J_0^n) \subset I_k$ for all $k = 1, 2, ..., n-1$. Let $J = J_0^n$; then the statement holds as formulated in the lemma. ☐

Lemma 3. *If $I_0 \rightharpoonup I_1 \rightharpoonup I_2 \rightharpoonup \ldots \rightharpoonup I_{n-1} \rightharpoonup I_0$, then there exists an $x \in I_0$ such that $x = f^n(x)$ and $f^k(x) \in I_k$ for $k = 0, 1, ..., n-1$.*

Proof. By lemma 2, there exists a closed interval $J \subset I_0$ such that $f^n(J) = I_0$ and $f^k(J) \subset I_k$ for $k = 1, 2, ..., n-1$ (at $k = 0$, $f^0(J) = J \subset I_0$ holds trivially). Then, by lemma 1, the map f^n has a fixed point $x \in J \subset I_0$. Also, since $x \in J$, $f^k(x) \in f^k(J) \subset I_k$ for all $k = 0, 1, ..., n-1$. ☐

If we consider a collection $\{I_0, I_1, ...\}$ of closed intervals $I_i \subset I$ with pairwise disjoint interiors, the covering relations $\rightharpoonup$ can be used to construct a directed graph between the intervals, called a Markov graph of f associated to $\{I_0, I_1, ...\}$. By lemma 3, any loop in the Markov graph generates a periodic orbit of f.

Now, let us prove Li & Yorke's theorem.

Proof. Consider the period-3 orbit $\{p_1, p_2, p_3\}$, $p_2 = f(p_1)$, $p_3 = f(p_2)$, $p_1 = f(p_3)$. Without loss of generality, we can assume that $p_1 < p_2 < p_3$. Then, we can define the intervals $I_1 = [p_1, p_2]$, $I_2 = [p_2, p_3]$. By construction and the continuity of f, $I_1 \rightharpoonup I_2$, and $I_2 \rightharpoonup I_1 \cup I_2$.

Thus, for any integer $n \geq 1$, we can construct a chain of covering relations of the form $I_1 \rightharpoonup I_2 \rightharpoonup I_2 \rightharpoonup \ldots \rightharpoonup I_2 \rightharpoonup I_1$ - that is, a closed loop of the Markov graph of f, containing $n-1$ occurrences of I_2. By lemma 5, the map f^n has a fixed point $x \in I_1$, and thus there exists a cycle of period n.

What we still need to show is that n is indeed the minimal period of the generated cycle. Assume that the orbit of the fixed point x of f^n is a cycle of minimal period $k < n$, $n \geq 2$. Then necessarily $x \in I_1 \cap I_2$, since $x \in I_1$, but also $x = f^k(x) \in I_2$, by construction. Thus we must have $x \in I_1 \cap I_2 = \{p_2\}$, for a minimal period of $k < n$ to be possible. By the uniqueness of orbits, there is a single orbit passing through p_2, the period-3 cycle we assume by construction, and therefore $k = 3$. This immediately excludes the possibility that the cycle has minimal period $k < n$ for $n = 2$. Take $n > 2$, then by construction we must have $f^2(x) \in I_2$. But $f^2(x) = f^2(p_2) = p_1 \notin I_2$, yielding a contradiction. Thus, the generated cycle cannot be of minimal period $k < n$, concluding the proof.

If $p_1 < p_3 < p_2$, the only other possible case, we can define $I_1 = [p_3, p_2]$, $I_2 = [p_1, p_3]$, and the same argument follows. Thus the assumption $p_1 < p_2 < p_3$ is indeed without loss of generality. □

4 Cycle of period 3 of the logistic map

In this section, we study the logistic mapping [3],

$$x \mapsto f_\alpha(x) = f(x, \alpha) = \alpha x(1 - x), \quad x \in [0, 1].$$

We prove that at $\alpha_0 = 1 + 2\sqrt{2}$, the third iterate of the logistic mapping exhibits a fold bifurcation, generating a stable period-3 cycle and an unstable period-3 cycle as α increases.

This result allows us to apply Li & Yorke's theorem, and conclude that, at least for α close to α_0, the system generated by the logistic mapping has cycles of all periods. Thus, this section provides an application of Li & Yorke's theorem, and an example of a discrete-time dynamical system with infinitely many cycles.

Proof. First, notice that $f_\alpha(x)$, and thus all its iterates, are polynomials, and thus smooth in both arguments. Let us introduce the function $G(x, \alpha) = f_\alpha^3(x)$.

Then

$$G(x, \alpha) = \alpha^3 x(1 - x)(1 - \alpha x + \alpha x^2)(1 - \alpha^2 x + (\alpha^3 + \alpha^2)x^2 - 2\alpha^3 x^3 + \alpha^3 x^4).$$

From the conditions for a fold bifurcation discussed in section 2, for the third iterate $G(x, \alpha)$ to exhibit a fold bifurcation at α_0, we need to find a fixed point x_0 of G at α_0 (i.e., satisfying $G(x_0, \alpha_0) = x_0$) of period 3, such that

$$\begin{cases} G_x(x_0, \alpha_0) = 1, \\ G_{xx}(x_0, \alpha_0) \neq 0, \\ G_\alpha(x_0, \alpha_0) \neq 0. \end{cases}$$

We are tasked to prove that such a bifurcation occurs at $\alpha_0 = 1 + 2\sqrt{2}$. First, note that any pair (x, α) satisfying the above conditions must necessarily solve

$$\begin{cases} G(x, \alpha) - x = 0, \\ G_x(x, \alpha) - 1 = 0. \end{cases}$$

Eliminating x from this system (see appendix) yields a condition on α of the form

$$(\alpha^2 - 2\alpha - 7)(\alpha - 1)^2(\alpha^2 + \alpha + 1)^2(\alpha^2 - 5\alpha + 7)^2 = 0,$$

which has real solutions $\alpha_0 = 1+2\sqrt{2}$, $\alpha_1 = 1-2\sqrt{2}$, $\alpha_2 = 1$. Thus, indeed, $\alpha_0 = 1+2\sqrt{2}$ is a critical value of the parameter that solves the first two bifurcation conditions.

Then, let us find a fixed point x_0 of G at α_0. The fixed point condition $G(x, \alpha_0) - x = 0$ reduces to

$$x(7x - 8 + 2\sqrt{2})(343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2}) = 0,$$

(see appendix). The function $G(x, \alpha_0)$ thus has two trivial fixed points at α_0, $x = 0$, and $x = \frac{8-2\sqrt{2}}{7}$, and three other fixed points that correspond to the roots of the third-degree polynomial in the third factor.

Notice that $x = 0$ and $x = \frac{8-2\sqrt{2}}{7}$ are also fixed points of the function $f(x, \alpha_0)$ itself (see appendix), and thus period-1 points. Let us focus on the other three solutions, denoted $x_0 < x_1 < x_2$ which are by construction period-3 points, and must thus span a period-3 cycle.

The exact forms of x_0, x_1, x_2 are given in the appendix. Let us take $x_0 \approx 0.1599$. Then, by construction, x_0 is a fixed point of G at α_0, and satisfies $G_x(x_0, \alpha_0) = 1$. Furthermore, numerical calculations (again shown in the appendix) verify that indeed $G_{xx}(x_0, \alpha_0) \approx 177.94 \neq 0$, and $G_\alpha(x_0, \alpha_0) \approx -0.7794 \neq 0$, meaning that the non-degeneracy and transversality conditions, respectively, are also satisfied.

Thus, the third iterate of f indeed exhibits a fold bifurcation at $\alpha_0 = 1 + 2\sqrt{2}$. Since the quantity $-G_{xx}(x_0, \alpha_0)/G_\alpha(x_0, \alpha_0) > 0$, the two period-3 cycles, one stable and one unstable, appear for $\alpha > \alpha_0$. □

Thus, for α greater than but close to α_0, we can conclude that the system has a period-3 cycle, and thus, by Li & Yorke's theorem, also cycles of minimal period n for all $n \geq 1$ (and thus infinitely many cycles).

5 More on the logistic mapping

While the fold bifurcation at $\alpha = 1+2\sqrt{2}$ was of particular interest for the present paper, due to it creating a period-3 cycle, the logistic mapping exhibits a lot of other interesting types of behavior. This section traces the qualitative changes in the discrete-time system generated by the logistic mapping for $0 \leq \alpha \leq 4$, with the hope of better contextualizing the fold bifurcation shown above. [1] [2]

$0 \leq \alpha < 1$

For $\alpha = 0$, any $n > 1$ iterate of f is zero, so this case is not particularly interesting. For $0 < \alpha < 1$, it can be easily verified (see appendix) that the system only has one fixed point in $[0, 1]$, at $x = 0$. This equilibrium is stable (also see appendix). The convergence to this steady state is monotonic, as the graph of $f(x, \alpha)$ lies strictly below the diagonal.

$1 \leq \alpha < 3$

At $\alpha = 1$, a transcritical bifurcation occurs, at which the equilibrium at $x = 0$ loses stability, and a second, stable, fixed point appears in $[0, 1]$ at $x = 1 - \frac{1}{\alpha}$ (see appendix).

For $1 < \alpha \leq 2$, there are thus two fixed points, one unstable at $x = 0$, and one stable at $x = 1 - \frac{1}{\alpha}$. All orbits will converge monotonically to this latter fixed point. For $2 < \alpha < 3$, the system is qualitatively the same, but the convergence to the fixed point will no longer be monotonic, i.e., the orbit will oscillate around the steady state while converging.

$3 \leq \alpha < 1 + \sqrt{6} \approx 3.44949...$

At $\alpha = 3$, the stable fixed point at $x = 1 - \frac{1}{\alpha}$ undergoes a period-doubling bifurcation, i.e., loses stability, and a stable period-2 cycle is created (see appendix). Thus, for $3 < \alpha < 1 + \sqrt{6}$, almost all initial values will get attracted to the stable period-2 cycle.

$3.44949... \leq \alpha \leq 3.56994...$

At $\alpha = 1 + \sqrt{6}$, the stable period-2 cycle undergoes a period doubling bifurcation, i.e., loses stability, and a period-4 cycle is created. Then, at $\alpha \approx 3.54409...$, the period-4 cycle undergoes a period doubling bifurcation itself, and a stable period-8 cycle is created. Thus, for this range of α, we observe a period-doubling cascade. Its limit value is $\alpha \approx 3.56994...$ i.e., the periodic cycles continue to bifurcate with increasing frequency until that point. In this limit, there exists a periodic orbit of infinite period (an aperiodic orbit) called the Feigenbaum attractor. Thus, in this limit, the system becomes chaotic.

$3.56994... < \alpha \leq 4$

Past $\alpha = 3.56994...$, we observe chaotic dynamics, interrupted by windows of stability, characterized by the existence of a stable periodic orbit of some period k. These begin as the system is gradually attracted, under variation of α, into an orbit of a certain period, i.e., the system undergoes a fold bifurcation of some period n. As this coalescence into a periodic orbit happens, i.e., for α slightly smaller than the given bifurcation value, we observe "laminar" periodic behavior, interrupted by chaotic bursts. Then, past the fold bifurcation, the generated stable cycle again enters a period-doubling cascade, until a new attracting aperiodic orbit is generated. The fold bifurcation discussed in the previous section is exactly the beginning of one such window of stability.

In these windows, the chaotic dynamics do not disappear. However, this is not generally observed in simulations, since "most" initial conditions nevertheless get attracted to the given stable periodic orbit.

Finally, when $\alpha = 4$, dynamics are again chaotic. We can observe orbits of minimal period n for all $n \geq 1$, but these are all unstable; no stable periodic orbits are present in the system.

6 Conclusion

This paper aimed to provide a comprehensive proof of Li & Yorke's seminal theorem "period-3 implies chaos", prove the existence of a period-3 cycle in the logistic mapping at a certain parameter value, and provide the context to hopefully help the reader better appreciate these results. While a lot more could be said about both the logistic mapping and the order of implication of the existence of periodic orbits (called the Sharkovsky ordering [3]), it is nevertheless the hope of the author that this paper could be a good starting point for further study, and that it has sparked your interest about chaotic dynamics. Good luck!

Appendix

Fold bifurcation of the logistic mapping

Derivation of $G(x,\alpha)$

$$\begin{aligned}
f^1_{(\alpha)}(x) &= \alpha x(1-x),\\
f^2_{(\alpha)}(x) &= \alpha^2 x(1-x)(1-\alpha x+\alpha x^2),\\
G(x,\alpha) = f^3_{(\alpha)}(x) &= \alpha^3 x(1-x)(1-\alpha x+\alpha x^2)(1-\alpha^2 x+(\alpha^3+\alpha^2)x^2-2\alpha^3x^3+\alpha^3x^4).
\end{aligned}$$

Derivation of the critical values of α

The expanded form of $G(x,\alpha)$ is

$$\begin{aligned}
G(x,\alpha) = &-\alpha^7x^8+4\alpha^7x^7-(6\alpha^7+2\alpha^6)x^6+(4\alpha^7+6\alpha^6)x^5-(\alpha^7+6\alpha^6+\alpha^5+\alpha^4)x^4\\
&+(2\alpha^6+2\alpha^5+2\alpha^4)x^3-(\alpha^5+\alpha^4+\alpha^3)x^2+\alpha^3x.
\end{aligned}$$

Then

$$\begin{aligned}
G_x(x,\alpha) = &-8\alpha^7x^7+28\alpha^7x^6-6(6\alpha^7+2\alpha^6)x^5+5(4\alpha^7+6\alpha^6)x^4-4(\alpha^7+6\alpha^6+\alpha^5+\alpha^4)x^3\\
&+3(2\alpha^6+2\alpha^5+2\alpha^4)x^2-2(\alpha^5+\alpha^4+\alpha^3)x+\alpha^3.
\end{aligned}$$

For a fold bifurcation, we need that $G(x,\alpha)-x=0$, $G_x(x,\alpha)-1=0$. Let us sequentially eliminate the powers of x from this system of equations. For the values of (x,α) that satisfy these two conditions,

$$\begin{aligned}
0 = G_1(x,\alpha) =& x(G_x(x,\alpha)-1)-8(G(x,\alpha)-x)\\
=&0-4\alpha^7x^7+2(6\alpha^7+2\alpha^6)x^6-3(4\alpha^7+6\alpha^6)x^5+4(\alpha^7+6\alpha^6+\alpha^5+\alpha^4)x^4\\
&-5(2\alpha^6+2\alpha^5+2\alpha^4)x^3+6(\alpha^5+\alpha^4+\alpha^3)x^2-7(\alpha^3-1)x,\\
0 = G_2(x,\alpha) =& 2G_1(x,\alpha)-(G_x(x,\alpha)-1)\\
=&0+(-4\alpha^7+8\alpha^6)x^6+(12\alpha^7-24\alpha^6)x^5+(-12\alpha^7+28\alpha^6+8\alpha^5+8\alpha^4)x^4\\
&+(4\alpha^7+4\alpha^6-16\alpha^5-16\alpha^4)x^3+(-6\alpha^6+6\alpha^5+6\alpha^4+12\alpha^3)x^2\\
&+(2\alpha^5+2\alpha^4-12\alpha^3+14)x,\\
0 = G_3(x,\alpha) =& 4\alpha^7xG_2(x,\alpha)+G_1(x,\alpha)\\
=&\ldots
\end{aligned}$$

Proceding in this manner until all powers of x are eliminated, we arrive at a condition, which can be written in the form

$$(\alpha^2-2\alpha-7)(\alpha-1)^2(\alpha^2+\alpha+1)^2(\alpha^2-5\alpha+7)^2=0.$$

Fixed poins

The fixed point condition (at $\alpha_0 = 1 + 2\sqrt{2}$) requires

$$\begin{aligned}
0 =& G(x, \alpha_0) - x \\
=& \left(1+2\sqrt{2}\right)^3 x\,(1-x)\left(1-\left(1+2\sqrt{2}\right)x+\left(1+2\sqrt{2}\right)x^2\right) \\
& \left(1-\left(1+2\sqrt{2}\right)^2 x+\left(1+2\sqrt{2}\right)^3 x^2-2\left(1+2\sqrt{2}\right)^3 x^3+\left(1+2\sqrt{2}\right)^2 x^2+\left(1+2\sqrt{2}\right)^3 x^4\right)-x
\end{aligned}$$

This expression can be factorized to

$$0 = Ax(7x - 8 + 2\sqrt{2})(343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2})^2,$$

where $A = \frac{(1+2\sqrt{2})^3-1}{(-8+2\sqrt{2})(31-41\sqrt{2})^2}$.

The condition is trivially equivalent to

$$x(7x - 8 + 2\sqrt{2})(343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2}) = 0.$$

It has two trivial solutions, $x = 0$ and $x = \frac{8-2\sqrt{2}}{7} \approx 0.7388$, and three other solutions, $x_0 < x_1 < x_2$, which are the roots of the third-degree polynomial above. They are given by

$$\begin{aligned}
x_0 =& \frac{1}{21}(10+\sqrt{2}) - \frac{\sqrt[3]{\frac{1}{2}\left(25-22\sqrt{2}+i\sqrt{43011-29700\sqrt{2}}\right)}}{3\sqrt[3]{7^2}} \\
&+ \frac{4\sqrt{2}-9}{3\sqrt[3]{\frac{7}{2}\left(25-22\sqrt{2}+i\sqrt{43011-29700\sqrt{2}}\right)}} \approx 0.1599,
\end{aligned}$$

$$\begin{aligned}
x_1 =& \frac{1}{21}(10+\sqrt{2}) + \frac{(1+i\sqrt{3})\sqrt[3]{\frac{1}{2}\left(25-22\sqrt{2}+i\sqrt{43011-29700\sqrt{2}}\right)}}{6\sqrt[3]{7^2}} \\
&+ \frac{(4\sqrt{2}-9)(1-i\sqrt{3})}{3\sqrt[3]{28\left(25-22\sqrt{2}+i\sqrt{43011-29700\sqrt{2}}\right)}} \approx 0.5144,
\end{aligned}$$

$$\begin{aligned}
x_2 =& \frac{1}{21}(10+\sqrt{2}) + \frac{(1-i\sqrt{3})\sqrt[3]{\frac{1}{2}\left(25-22\sqrt{2}+i\sqrt{43011-29700\sqrt{2}}\right)}}{6\sqrt[3]{7^2}} \\
&+ \frac{(4\sqrt{2}-9)(1+i\sqrt{3})}{3\sqrt[3]{28\left(25-22\sqrt{2}+i\sqrt{43011-29700\sqrt{2}}\right)}} \approx 0.9563.
\end{aligned}$$

Verification that $x = 0$ and $x = \frac{8-2\sqrt{2}}{7}$ are period-1 points

Let us show that $x = 0$ and $x = \frac{8-2\sqrt{2}}{7}$ are also fixed points of f at α_0, and thus period-1 points. By definition, such points need to satisfy

$$f(x, \alpha_0) = (1 + 2\sqrt{2})x(1 - x) = x,$$

and we can indeed verify that

$$
\begin{aligned}
f(0, 1+2\sqrt{2}) &= (1+2\sqrt{2})0(1-0) = 0, \\
f\left(\frac{8-2\sqrt{2}}{7}, 1+2\sqrt{2}\right) &= (1+2\sqrt{2})\frac{8-2\sqrt{2}}{7}\left(1-\frac{8-2\sqrt{2}}{7}\right) = \frac{8-2\sqrt{2}}{7}.
\end{aligned}
$$

Fold bifurcation non-degeneracy and transversality conditions

From our function $G(x, \alpha)$, we obtain

$$
\begin{aligned}
G_{xx}(x,\alpha) &= -56\alpha^7x^6 + 168\alpha^7x^5 - 30(6\alpha^7 + 2\alpha^6)x^4 + 20(4\alpha^7 + 6\alpha^6)x^3 - 12(\alpha^7 + 6\alpha^6 + \alpha^5 + \alpha^4)x^2 \\
&\quad + 6(2\alpha^6 + 2\alpha^5 + 2\alpha^4)x - 2(\alpha^5 + \alpha^4 + \alpha^3), \\
G_{\alpha}(x,\alpha) &= -7\alpha^6x^8 + 28\alpha^6x^7 - (42\alpha^6 + 12\alpha^5)x^6 + (28\alpha^6 + 36\alpha^5)x^5 - (7\alpha^6 + 36\alpha^5 + 5\alpha^4 + 4\alpha^3)x^4 \\
&\quad + (12\alpha^5 + 10\alpha^4 + 8\alpha^3)x^3 - (5\alpha^4 + 4\alpha^3 + 3\alpha^2)x^2 + 3\alpha^2 x.
\end{aligned}
$$

At (x_0, α_0), these evaluate to

$$
\begin{aligned}
G_{xx}(x_0, \alpha_0) &\approx 177.82, \\
G_{\alpha}(x_0, \alpha_0) &\approx -0.7794.
\end{aligned}
$$

Thus, the non-degeneracy conditions of the fold bifurcation are satisfied.

Additional analysis

Equilibria and stability

As shown above, the fixed points of $f(x, \alpha)$ need to satisfy $f(x, \alpha) = x$, i.e. $\alpha x(1-x) = x$. Thus, we can find such points at $x = 0$, which exists independent of α, and $x = 1 - \frac{1}{\alpha}$, which is only in $[0, 1]$ if $\alpha \geq 1$.

Thus, for $0 \leq \alpha < 1$, we only have one fixed point in $[0, 1]$, at $x = 0$. It is stable, since $|f_x(0, \alpha)| = |\alpha - 2\alpha \times 0| = |\alpha| < 1$. The other fixed point, at $x = 1 - \frac{1}{\alpha}$, only exists outside the interval $[0, 1]$, so it is technically not a fixed point of the system. Nevertheless, we can extend f to $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and find its stability; since $|f_x(x, \alpha)| = |\alpha - 2\alpha(1 - \frac{1}{\alpha})| = |-\alpha + 2| > 1$, the fixed point is unstable.

At $\alpha = 1$, the second fixed point $x = 1 - \frac{1}{\alpha}$ enters $[0, 1]$ and the two fixed points coincide at $x = 0$. Thus, we can expect a transcritical bifurcation, at which the two fixed points collide and exchange stability. Indeed, we see that at $\alpha = 1$, $|f_x(0, 1)| = 1$. For $1 < \alpha < 3$, the $x = 0$ fixed point becomes unstable ($|f_x(0, \alpha)| = |\alpha| > 1$), and the $x = 1 - \frac{1}{\alpha}$ fixed point becomes stable ($|f_x(x, \alpha)| = |-\alpha + 2| < 1$).

More formally, we can see that the conditions for a transcritical bifurcation are satisfied at $\alpha = 1$, $x = 0$:

$$f(0,1) = 1 \times 0 \times (1-0) = 0$$
$$f_x(0,1) = 1 - 2 \times 1 \times 0 = 1$$
$$f_{\alpha x}(0,1) = [1-2x]_{(0,1)} = 1 \neq 0$$
$$f_{xx}(0,1) = [-2\alpha]_{(0,1)} = -2 \neq 0$$

Period-doubling bifurcation

At $\alpha = 3$, the $x = 1 - \frac{1}{\alpha}$ fixed point undergoes another stability change, but this time in the context of a period-doubling bifurcation. We can indeed verify that $f_x(1-1/3, 3) = -3+2 = -1$, and furthermore, $\alpha = [f_{\mu x} + \frac{1}{2}(f_\mu)(f_{xx})]_{(2/3,3)} = (1-2\times\frac{2}{3}) + \frac{1}{2}(\frac{1}{9})(-6) = -\frac{2}{3} \neq 0$, and $\beta = (\frac{1}{3!}f_{xxx}(2/3,3)) + (\frac{1}{2!}f_{xx}(2/3,3))^2 = 0 + (\frac{1}{2}\times(-6))^2 = 9 \neq 0$. Based on these results, and specifically, since $-\beta/\alpha > 0$, we can also see that the stable period-2 orbit exists for α close to, but greater than 3.

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