Uniqueness of limit cycles near the Bogdanov-Takens Bifurcation.

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We take a look at the Bogdanov normal form and its limit cycles:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= \beta_1 + \beta_2 x + x^2 - xy. \end{cases} \tag{1}$$

In this essay, we translate this system into an orbitally equivalent one. Then we use this new system to prove that, for parameters between curves[1] H and P, we have a unique limit cycle. Lastly, we look at the full bifurcation diagram near this bifurcation.

First, we look at the equilibria of system (1) given by setting $\dot{x} = 0$ and $\dot{y} = 0$. We immediately see $y = 0$ and as a consequence $x = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4\beta_1}}{2}$. This gives us a restriction, as we want the equilibria to be real, we see that $\beta_2^2 \geq 4\beta_1$. So our equilibria are

$$(x, y) = \left(\frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4\beta_1}}{2}, 0 \right) \quad \text{for } \beta_2^2 \geq 4\beta_1. \tag{2}$$

1 Orbital equivalence

Now, we show that this normal form is orbitally equivalent to a perturbed Hamiltonian system of the form

$$\begin{cases} \dot{u} &= v, \\ \dot{v} &= u(u-1) - (\gamma_1 v + \gamma_2 uv), \end{cases} \tag{3}$$

with $\gamma_j = \gamma_j(\beta) \to 0$ as $\beta \to 0$.

We start by introducing an intermediary change of variables $x = r + \alpha$ with $\alpha = \frac{-\beta_2 - \sqrt{\beta_2^2 - 4\beta_1}}{2}$. As shorthand, we use $\mu = \sqrt{\beta_2^2 - 4\beta_1}$. We see that $\dot{x} = \dot{r} = y$ and for $\dot{y}$ we get:

$$\begin{aligned} \dot{y} &= \beta_1 + \beta_2 r + \alpha\beta_2 + (r + \alpha)^2 - y(r + \alpha) \\ &= \beta_1 + \beta_2 r + \alpha\beta_2 + r^2 + 2\alpha r + \alpha^2 - ry - \alpha y \\ &= \beta_1 + \beta_2 r - \frac{\beta_2^2}{2} - \frac{\beta_2 \mu}{2} + r^2 - \beta_2 r - \mu r + \frac{\beta_2^2}{4} + \frac{\beta_2 \mu}{2} + \frac{\mu^2}{4} - ry - \frac{\beta_2 y}{2} - \frac{\mu y}{2} \\ &= \beta_1 - \frac{\beta_2^2}{4} + r^2 - \mu r + \frac{\beta_2^2}{4} - \beta_1 - ry - \frac{\beta_2 y}{2} - \frac{\mu y}{2} \\ &= r^2 - \mu r - ry - \frac{\beta_2 y}{2} - \frac{\mu y}{2} \\ &= r(r - \mu) - (\alpha y + ry). \end{aligned} \tag{4}$$

[1] These curves are determined later in this essay.

Now we use another change of variables and a reparametrization of time by taking $u(\tau) = \frac{r}{\mu}$, $v(\tau) = \frac{y}{\mu^{\frac{3}{2}}}$ and $t = \frac{\tau}{\mu^{\frac{1}{2}}}$. We then get the following equations:

$$\frac{du}{d\tau} = \frac{1}{\mu}\frac{1}{\mu^{\frac{1}{2}}}\frac{dr}{dt} = \frac{1}{\mu^{\frac{3}{2}}}y = \frac{1}{\mu^{\frac{3}{2}}}\mu^{\frac{3}{2}}v = v \tag{5}$$

and

$$\begin{aligned}\frac{dv}{d\tau} &= \frac{1}{\mu^{\frac{3}{2}}}\frac{1}{\mu^{\frac{1}{2}}}\frac{dy}{dt} \\ &= \frac{1}{\mu^2}\left[\mu u(\mu u - \mu) - (\alpha\mu^{\frac{3}{2}}v + \mu u\mu^{\frac{3}{2}}v)\right] \\ &= u(u-1) - (\alpha\mu^{-\frac{1}{2}}v + \mu^{\frac{1}{2}}uv) \\ &= u(u-1) - (\gamma_1 v + \gamma_2 uv).\end{aligned} \tag{6}$$

In the last step we took $\gamma_1 = \alpha\mu^{-\frac{1}{2}} = -\frac{1}{2}\left(\beta_2 + \sqrt{\beta_2^2 - 4\beta_1}\right)(\beta_2^2 - 4\beta_1)^{-\frac{1}{4}}$ and $\gamma_2 = \mu^{\frac{1}{2}} = (\beta_2^2 - 4\beta_1)^{\frac{1}{4}}$. We then see that if $\beta \to 0$ then $\gamma_j \to 0$. Now our system looks like.

$$\begin{cases}\dot{u} &= v, \\ \dot{v} &= u(u-1) - (\gamma_1 v + \gamma_2 uv).\end{cases} \tag{7}$$

We have now shown that system (1) is orbitally equivalent to system (3) by using only special equivalences from section 2.2 of [2]. We also see that this is a perturbed Hamiltonian system with Hamiltonian $\mathcal{H}(u,v) = \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{1}{3}u^3$. If we look at what these transformations did to the equilibria of system 1, we see that the v coordinate is 0, and the u coordinates have become

$$u = \frac{x-\alpha}{\mu} = \frac{1}{\mu}\left(\frac{-\beta_2 \pm \mu}{2} - \frac{-\beta_2 - \mu}{2}\right) = 0 \text{ or } 1. \tag{8}$$

Thus, we see that these transformations have fixed the equilibria in place at $(u,v) = (0,0)$ and $(1,0)$ in the new system.

2 Uniqueness of limit cycle

First, we look at the unperturbed system. This is a Hamiltonian system with $\mathcal{H}(u,v) = \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{1}{3}u^3$ and our equilibria occur at $(u,v) = (0,0)$ and $(1,0)$. The Jacobi matrix is given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 2u-1 & 0 \end{pmatrix}. \tag{9}$$

At the point $(1,0)$, we have $\lambda_{1,2} = \pm 1$. This constitutes an unstable saddle. At point $(0,0)$ we then have $\lambda_{1,2} = \pm i$. This constitutes an elliptic equilibrium or, as it is usually called in a Hamiltonian system, a center. Now we want to see what happens to the Hamiltonian $\mathcal{H}$ over time. By calculating $\dot{\mathcal{H}}$ we have

$$\begin{aligned}\dot{\mathcal{H}} &= v\dot{v} + u\dot{u} - u^2\dot{u} \\ &= v(u^2 - u) + uv - u^2 v \\ &= 0\end{aligned} \tag{10}$$

So we see that $\mathcal{H}$ is a conserved quantity in our system. Thus, these orbits do not change over time. We see that the level curves $L_h := \{(u,v) : \mathcal{H} = h, h \in [0, \frac{1}{6}]\}$ for the Hamiltonian stay bounded when looking at $u \in [-\frac{1}{2}, 1]$. By Poincaré-Bendixson, we then have homoclinic periodic orbits for $h \in (0, \frac{1}{6})$ and a union of a periodic orbit and an equilibrium at $h = \frac{1}{6}$.

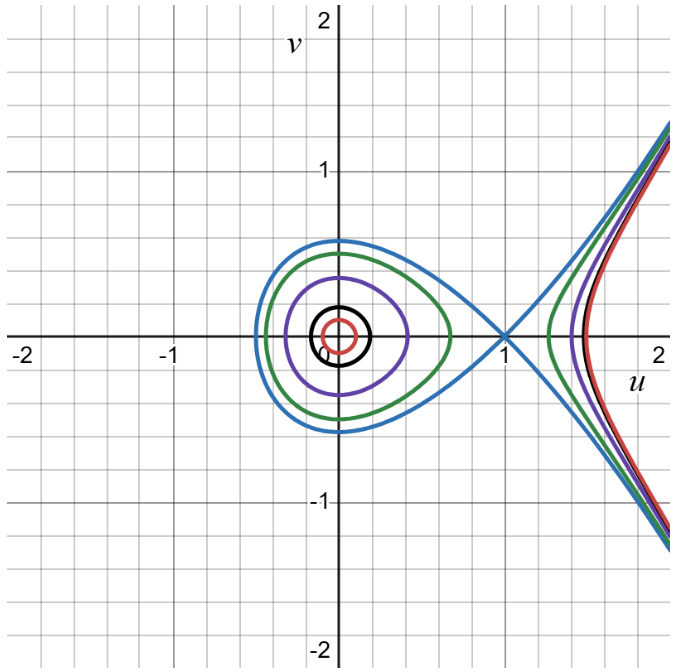


Figure 1: Level curves for $\mathcal{H}(u,v) = h$ with $h = 0.005$ (red), $\frac{1}{64}$ (black), $\frac{1}{16}$ (purple), $\frac{1}{8}$ (green), and $\frac{1}{6}$ (blue).

2.1 Periodic orbits of the perturbed system

To prove there is a unique limit cycle, we will introduce a helper function

$$M(h,\gamma) = \oint_{L_h} (\gamma_1 v + \gamma_2 uv)du. \tag{11}$$

Now when we set $M(h,\gamma) = 0$ and have

$$\begin{aligned} M(h,\gamma) &= \oint_{L_h} (\gamma_1 v + \gamma_2 uv)du \\ 0 &= \gamma_1 \oint_{L_h} vdu + \gamma_2 \oint_{L_h} uvdu \\ -\frac{\gamma_1}{\gamma_2} &= \frac{\oint_{L_h} uvdu}{\oint_{L_h} vdu}. \end{aligned} \tag{12}$$

We now define some shorthand notation $I_0(h) = \oint_{L_h} vdu$ and $I_1(h) = \oint_{L_h} uvdu$ and $Q(h) = \frac{I_1(h)}{I_0(h)}$. Thus we have $Q(h) = -\frac{\gamma_1}{\gamma_2}$. In [1] it is proven that it is sufficient to show that $M(h,\gamma) = 0$ gives restricting curves for small $||\gamma||$ at the boundaries of the interval $\left[0, \frac{1}{6}\right]$ and that $Q(h)$ is monotone on this interval. Now we can directly calculate those restricting curves by solving I_1 and I_0 for those values.

2.2 Level curve $h \to 0$

For $h = 0$, we see that the orbit of $\mathcal{H}(u,v)$ collapses to a single point in figure (1). As $h \to 0$, we can then argue that the quadratic terms become leading, giving us $\mathcal{H}(u,v) \approx \frac{1}{2}(v^2 + u^2)$. This is a circle with a radius of $r = \sqrt{2h}$. We can now take the substitution $u = \sqrt{2h}\cos\theta$ and $v = \sqrt{2h}\sin\theta$. Then $du = -\sqrt{2h}\sin\theta d\theta$ and our closed orbit becomes a full rotation around this circle. Filling this into $I_0(h)$, we get

$$\begin{aligned} I_0(h) &= \int_0^{2\pi} \sqrt{2h}\sin\theta(-\sqrt{2h}\sin\theta)d\theta \\ &= -2h \int_0^{2\pi} \sin^2\theta d\theta \\ &= -2\pi h. \end{aligned} \tag{13}$$

Doing the same to $I_1(h)$, we get

$$\begin{aligned} I_1(h) &= \int_0^{2\pi} \sqrt{2h}\cos\theta\sqrt{2h}\sin\theta(-\sqrt{2h}\sin\theta)d\theta \\ &= -(2h)^{\frac{3}{2}}\int_0^{2\pi}\cos\theta\sin^2\theta d\theta \\ &= 0. \end{aligned} \tag{14}$$

Thus, we see that $\lim_{h\to 0} Q(h) = 0$. This means that we have the restriction $-\frac{\gamma_1}{\gamma_2} = 0$ and thus a restrictive curve

$$H = \{\gamma : \gamma_1 = 0, \gamma_2 > 0\}.$$

Here we derive the condition $\gamma_2 > 0$ from the condition[2] $\beta_2^2 > 4\beta_1$.

2.3 Level curve $h \to \frac{1}{6}$

For $h = \frac{1}{6}$ we see that $\mathcal{H}(u,v) = \frac{1}{6}$ gives us the following curve: $v = \pm\sqrt{\frac{1}{3} - u^2 + \frac{2}{3}u^3}$. This creates a closed orbit between $u = -\frac{1}{2}$ and $u = 1$. Now, in our calculation of I_1 and I_0, we need to account for the two branches of our equation for v. These run in opposite directions, so we get

$$\begin{aligned} I_0(1/6) &= \int_{-\frac{1}{2}}^{1}\sqrt{\frac{1}{3} - u^2 + \frac{2}{3}u^3}du - \int_{1}^{-\frac{1}{2}}\sqrt{\frac{1}{3} - u^2 + \frac{2}{3}u^3}du \\ &= 2\int_{-\frac{1}{2}}^{1}\sqrt{\frac{1}{3} - u^2 + \frac{2}{3}u^3}du \\ &= \frac{6}{5}. \end{aligned} \tag{15}$$

The last step is done by using an integrator like Wolfram Alpha. For $I_1(h)$ we get

$$\begin{aligned} I_1(1/6) &= \int_{-\frac{1}{2}}^{1} u\sqrt{\frac{1}{3} - u^2 + \frac{2}{3}u^3}du - \int_{1}^{-\frac{1}{2}} u\sqrt{\frac{1}{3} - u^2 + \frac{2}{3}u^3}du \\ &= 2\int_{-\frac{1}{2}}^{1} u\sqrt{\frac{1}{3} - u^2 + \frac{2}{3}u^3}du \\ &= \frac{6}{35}. \end{aligned} \tag{16}$$

This means that we have $\lim_{h\to\frac{1}{6}} Q(h) = \frac{6/5}{6/35} = \frac{1}{7}$. This gives us $\frac{1}{7} = -\frac{\gamma_1}{\gamma_2}$ and thus the restrictive curve

$$P = \{\gamma : \gamma_1 = -\frac{1}{7}\gamma_2, \gamma_2 > 0\}.$$

2.4 Monotonicity of $Q(h)$

Now we prove the monotonicity of $Q(h)$. First, we show that I_0 and I_1 satisfy the Picard-Fuchs Equations

$$\begin{cases} h\left(h - \frac{1}{6}\right)I_0'(h) = \left(\frac{5}{6}h - \frac{1}{6}\right)I_0(h) + \frac{7}{36}I_1(h), \\ h\left(h - \frac{1}{6}\right)I_1'(h) = -\frac{1}{6}hI_0(h) + \frac{7}{6}hI_1(h). \end{cases} \tag{17}$$

Then, we show that $Q(h)$ satisfies the Riccati equation

$$h\left(h - \frac{1}{6}\right)Q'(h) = -\frac{7}{36}Q(h)^2 + \left(\frac{h}{3} + \frac{1}{6}\right)Q(h) - \frac{h}{6}. \tag{18}$$

Lastly, we prove that for $h \in (0, \frac{1}{6})$ we have $0 \le Q(h) \le \frac{1}{7}$ and that $Q'(h) > 0$ for all $h \in [0, \frac{1}{6}]$.

[2] Here we left out $\beta_2^2 = 4\beta_1$ as this constitutes the Bogdanov-Takens Bifurcation.

2.4.1 Picard-Fuchs equation

We see that the level curves L_h give us a representation of v in terms of u and h. If we now take the derivative with respect to h of $\mathcal{H}$, we have

$$\frac{dv}{dh} = \frac{1}{v}. \tag{19}$$

We use these facts to calculate $\frac{dI_0}{dh}$ and $\frac{dI_1}{dh}$. Using the Leibniz Integral Rule with $v(u,h) = \pm\sqrt{2h - u^2 + \frac{2}{3}u^3}$ and with ends points $u_{1,2}$ such that $v(u_{1,2}, h) = 0$, we get

$$\begin{aligned}
\frac{dI_0}{dh} &= \frac{d}{dh}\oint_{L_h} v(u,h)du \\
&= \frac{d}{dh}2\int_{u_1}^{u_2} v(u,h)du \\
&= 2\left(v(u_2,h)\frac{du_2}{dh} - v(u_1,h)\frac{du_1}{dh} + \int_{u_1}^{u_2}\frac{\partial}{\partial h}vdu\right) \\
&= 2\int_{u_1}^{u_2}\frac{du}{v} \\
&= \oint_{L_h}\frac{du}{v}
\end{aligned} \tag{20}$$

$$\begin{aligned}
\frac{dI_1}{dh} &= \frac{d}{dh}\oint_{L_h} uvdu \\
&= 2\left(u_2v(u_2,h)\frac{du_2}{dh} - u_1v(u_1,h)\frac{du_1}{dh} + \int_{u_1}^{u_2}\frac{\partial}{\partial h}uvdu\right) \\
&= \oint_{L_h}\frac{udu}{v}
\end{aligned} \tag{21}$$

It is important to note that u does not depend on h but the end points $u_{1,2}$ do as the value of h determines where $v(u,h) = 0$. If we now take $\frac{dH}{du}$, we have $v\frac{dv}{du} + u - u^2 = 0$. Multiplying this by $\frac{u^m}{v}$, integrating over L_h and using integration by parts, we have

$$\begin{aligned}
0 &= \frac{u^m}{v}\left(v\frac{dv}{du} + u - u^2\right) \\
&= u^m\frac{dv}{du} + \frac{u^{m+1}}{v} - \frac{u^{m+2}}{v} \\
\oint_{L_h} 0\ du &= \oint_{L_h}\left(u^m\frac{dv}{du} + \frac{u^{m+1}}{v} - \frac{u^{m+2}}{v}\right)du \\
0 &= \oint_{L_h} u^m\frac{dv}{du}du + \oint_{L_h}\frac{u^{m+1}}{v}du - \oint_{L_h}\frac{u^{m+2}}{v} \\
\oint_{L_h}\frac{u^{m+2}}{v} &= \oint_{L_h} u^m\frac{dv}{du}du + \oint_{L_h}\frac{u^{m+1}}{v}du \\
&= u^mv\big|_{L_h} - m\oint_{L_h} u^{m-1}vdu + \oint_{L_h}\frac{u^{m+1}}{v}du \\
&= \oint_{L_h}\frac{u^{m+1}}{v}du - m\oint_{L_h} u^{m-1}vdu.
\end{aligned} \tag{22}$$

The part $u^m v\big|_{L_h}$ vanishes as the curve L_h has the same start and end point. We will use this identity for $m = 0, 1$ and 2. These values of m give us the following identities:

$$\begin{aligned} m = 0: \quad \oint_{L_h} \frac{u^2}{v} du &= \oint_{L_h} \frac{u}{v} du \\ &= \frac{dI_1}{dh}, \end{aligned} \tag{23}$$

$$\begin{aligned} m = 1: \quad \oint_{L_h} \frac{u^3}{v} du &= \oint_{L_h} \frac{u^2}{v} du - \oint_{L_h} v du \\ &= \frac{dI_1}{dh} - I_0, \end{aligned} \tag{24}$$

$$\begin{aligned} m = 2: \quad \oint_{L_h} \frac{u^4}{v} du &= \oint_{L_h} \frac{u^3}{v} du - 2\oint_{L_h} uv du \\ &= \frac{dI_1}{dh} - I_0 - 2I_1. \end{aligned} \tag{25}$$

Here we immediately used what we already calculated in the previous steps. Using equations (20), (21), (23), (24) and (25), we can now calculate $h\frac{dI_0}{dh}$ and $h\frac{dI_1}{dh}$. We have

$$\begin{aligned} h\frac{dI_0}{dh} &= h \oint_{L_h} \frac{du}{v} \\ &= \oint_{L_h} h \frac{du}{v} \quad (h \text{ is a constant}) \\ &= \oint_{L_h} \left(\frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{1}{3}u^3\right) \frac{du}{v} \\ &= \frac{1}{2}\oint_{L_h} v du + \frac{1}{2}\oint_{L_h} \frac{u^2}{v} du - \frac{1}{3}\oint_{L_h} \frac{u^3}{v} du \\ &= \frac{1}{2} I_0 + \frac{1}{2}\frac{dI_1}{dh} - \frac{1}{3}\left(\frac{dI_1}{dh} - I_0\right) \\ &= \frac{5}{6} I_0 + \frac{1}{6}\frac{dI_1}{dh}, \end{aligned} \tag{26}$$

$$\begin{aligned} h\frac{dI_1}{dh} &= \oint_{L_h} h\frac{u}{v} du \\ &= \oint_{L_h} \left(\frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{1}{3}u^3\right) \frac{u}{v} du \\ &= \frac{1}{2}\oint_{L_h} uv du + \frac{1}{2}\oint_{L_h} \frac{u^3}{v} du - \frac{1}{3}\oint_{L_h} \frac{u^4}{v} du \\ &= \frac{1}{2} I_1 + \frac{1}{2}\left(\frac{dI_1}{dh} - I_0\right) - \frac{1}{3}\left(\frac{dI_1}{dh} - I_0 - 2I_1\right) \\ &= \frac{7}{6} I_1 - \frac{1}{6} I_0 + \frac{1}{6}\frac{dI_1}{dh} \\ \left(h - \frac{1}{6}\right)\frac{dI_1}{dh} &= \frac{7}{6} I_1 - \frac{1}{6} I_0. \end{aligned} \tag{27}$$

Now we can substitute (27) into (26). We have

$$\begin{aligned} h\left(h - \frac{1}{6}\right)\frac{dI_0}{dh} &= \frac{5}{6}\left(h - \frac{1}{6}\right) I_0 + \frac{1}{6}\left(h - \frac{1}{6}\right)\frac{dI_1}{dh} \\ &= \left(\frac{5h}{6} - \frac{5}{36}\right) I_0 + \frac{1}{6}\left(\frac{7}{6} I_1 - \frac{1}{6} I_0\right) \\ &= \left(\frac{5h}{6} - \frac{1}{6}\right) I_0 + \frac{7}{36} I_1. \end{aligned} \tag{28}$$

This gives us the Picard-Fuchs equations (17).

2.4.2 Riccati equation

Using the Picard-Fuchs equations, we can now calculate $\frac{dQ}{dh}$. We get

$$\begin{aligned}\frac{dQ}{dh} &= \frac{I_0 I_1' - I_0' I_1}{I_0^2} \\ h\left(h-\frac{1}{6}\right)Q' &= \frac{1}{I_0^2}\left(I_0\left[-\frac{1}{6}hI_0+\frac{7}{6}hI_1\right]-I_1\left[\left(\frac{5}{6}h-\frac{1}{6}\right)I_0+\frac{7}{36}I_1\right]\right) \\ &= -\frac{h}{6}+\frac{7}{6}h\frac{I_1}{I_0}-\left(\frac{5}{6}h-\frac{1}{6}\right)\frac{I_1}{I_0}-\frac{7}{36}\frac{I_1^2}{I_0^2} \\ &= -\frac{7}{36}Q^2+\left(\frac{h}{3}+\frac{1}{6}\right)Q-\frac{h}{6}.\end{aligned} \tag{29}$$

This is the Riccati equation (18).

2.4.3 Bounds of $Q(h)$

As we are looking at small values of h, we know that the linear terms of Q will dominate. So we use the following Taylor approximation $Q = \eta h + \mathcal{O}(h^2)$ with $\eta \in \mathbb{R}$. Now we can find the value of η by substituting our approximation into the Riccati equation and ignoring the $\mathcal{O}(h^2)$ terms. We have

$$\begin{aligned}h\left(h-\frac{1}{6}\right)\eta &= -\frac{7}{36}\eta^2h^2+\left(\frac{h}{3}+\frac{1}{6}\right)\eta h-\frac{h}{6} \\ -\frac{1}{6}\eta h &= \frac{1}{6}\eta h-\frac{1}{6}h \\ 2\eta &= 1 \\ \eta &= \frac{1}{2}.\end{aligned} \tag{30}$$

So we see that $Q'(0) = \frac{1}{2} > 0$. We now show, by way of contradiction, that $0 \le Q(h) \le \frac{1}{7}$ for $h \in [0, \frac{1}{6}]$. Suppose $\bar{h} \in \left(0, \frac{1}{6}\right]$ is the first intersection of Q and the h-axis. So $Q(\bar{h}) = 0$ and $Q(h) > 0$ for $h \in (0, \bar{h})$. Then, from the Riccati equation, we can see that

$$\bar{h}\left(\bar{h}-\frac{1}{6}\right)Q'(\bar{h}) = -\frac{7}{36}Q(\bar{h})^2+\left(\frac{\bar{h}}{3}+\frac{1}{6}\right)Q(\bar{h})-\frac{\bar{h}}{6} = -\frac{\bar{h}}{6}. \tag{31}$$

Thus $(\bar{h}-\frac{1}{6})Q'(\bar{h}) = -\frac{1}{6}$. Now we see that $Q'(\bar{h}) > 0$ for any $\bar{h} \in (0, \frac{1}{6}]$ with $Q(\bar{h}) = 0$, as $(\bar{h}-\frac{1}{6})$ is negative. As we know $Q'(0) > 0$ and $Q(0) = 0$, Q is positive close to $h = 0$. For Q to then intersect the h-axis, $Q'(\bar{h})$ must be negative. This leads to a contradiction at the $h = 0$ boundary. Thus we have $Q(h) \ge 0$ for all $h \in [0, \frac{1}{6}]$. Now suppose $\bar{h} \in (0, \frac{1}{6})$ is a value such that $Q(\bar{h}) = \frac{1}{7}$ for the first time. Using the Riccati equation, we then see that

$$\begin{aligned}\bar{h}\left(\bar{h}-\frac{1}{6}\right)Q'(\bar{h}) &= -\frac{7}{36}Q(\bar{h})^2+\left(\frac{\bar{h}}{3}+\frac{1}{6}\right)Q(\bar{h})-\frac{\bar{h}}{6} \\ &= -\frac{1}{7\cdot 36}+\frac{1}{7\cdot 6}+\frac{\bar{h}}{7\cdot 3}-\frac{\bar{h}}{6} \\ &= \frac{5}{7\cdot 36}-\frac{5}{7\cdot 6}\bar{h} \\ &= \frac{5}{7\cdot 6}\left(\frac{1}{6}-\bar{h}\right), \\ \text{so } Q'(h) &= -\frac{5}{7\cdot 6\cdot \bar{h}} < 0\end{aligned} \tag{32}$$

As we know $Q(\frac{1}{6}) = \frac{1}{7}$ and $Q(0) = 0$, $Q(h)$ must be less than $\frac{1}{7}$ if $h \in (0, \bar{h})$. Then for Q to reach $\frac{1}{7}$, $Q'(\bar{h})$ must be positive. This gives us a contradiction and we see that $Q(h) < \frac{1}{7}$ for all $h \in [0, \frac{1}{6})$. Thus, we have $0 \le Q(h) \le \frac{1}{7}$ for $h \in [0, \frac{1}{6}]$. Now, we want to know if the extrema of $Q(h)$ are minima or maxima. We verify this by examining the derivative of the Riccati equation at $Q'(h) = 0$. We have

$$\begin{aligned}\left(2hQ' + h^2Q'' - \frac{1}{6}Q' - \frac{h}{6}Q''\right)\Big|_{Q'=0} &= \left(-\frac{14}{36}QQ' + \frac{1}{3}Q + \frac{h}{3}Q' + \frac{1}{6}Q' - \frac{1}{6}\right)\Big|_{Q'=0} \\ \left(h^2 - \frac{h}{6}\right)Q'' &= \frac{1}{3}Q - \frac{1}{6} \\ &\leq \frac{1}{3\cdot 7} - \frac{1}{6} \\ &= -\frac{5}{6\cdot 7} < 0\end{aligned} \tag{33}$$

So, as $h\left(h - \frac{1}{6}\right) < 0$, we see that $Q''(h) > 0$. This means that all points where $Q'(h) = 0$ are minima. So there can be at most one extremum. We also see that if $Q'(h) > 0$ for a certain h, the extremum must lie before that point. Now, as $Q'(0) > 0$, we see that $Q'(h)$ cannot be zero for any $h \in [0, \frac{1}{6}]$. Thus Q is monotonically increasing for $h \in [0, \frac{1}{6}]$. Thus, we can conclude that for $||\gamma||$ sufficiently small and between curves H and P, we have a unique limit cycle.

3 Bifurcation diagram

Now we can study the complete bifurcation diagram of our original (β_1, β_2). We describe what happens on the found curves and in the different regions. As system (3) is orbitally equivalent to system (1), we determine the bifurcations and phase portraits using this system.

We first examine the general case for system (3) and use that to define the different regions. We saw that the equilibria of system (3) are $(u, v) = (0, 0)$ or $(1, 0)$. The Jacobi matrix of system (3) is

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 2u - 1 - \gamma_2 v & -\gamma_1 - \gamma_2 u \end{pmatrix}. \tag{34}$$

For our $(1, 0)$ equilibrium, we have $\lambda_{1,2} = -\left(\frac{\gamma_1+\gamma_2}{2}\right) \pm \sqrt{\frac{(\gamma_1+\gamma_2)^2}{4} + 1}$. We can see that for all values of γ, except $||\gamma|| = 0$, this gives us a hyperbolic saddle. This is because $\lambda_1 > 0$ and $\lambda_2 < 0$. On $(0, 0)$ our eigenvalues become $\lambda_{1,2} = -\frac{\gamma_1}{2} \pm \sqrt{\frac{\gamma_1^2}{4} - 1}$. This equilibrium is what we are studying in the coming sections. From our section on orbital equivalence, we saw that the curve $T_\pm := \{\beta : \beta_2^2 = 4\beta_1, ||\beta|| \neq 0\}$ has special properties. From our section on the uniqueness of the limit cycle, we have seen that curves P and H also have special properties. This gives us the diagram in figure 2.

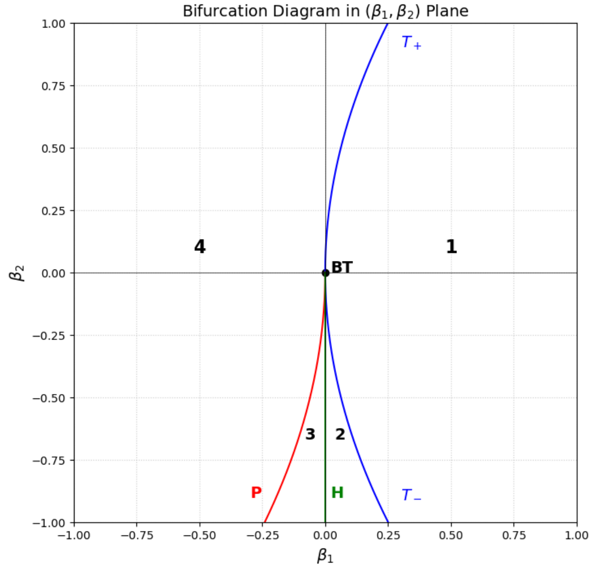
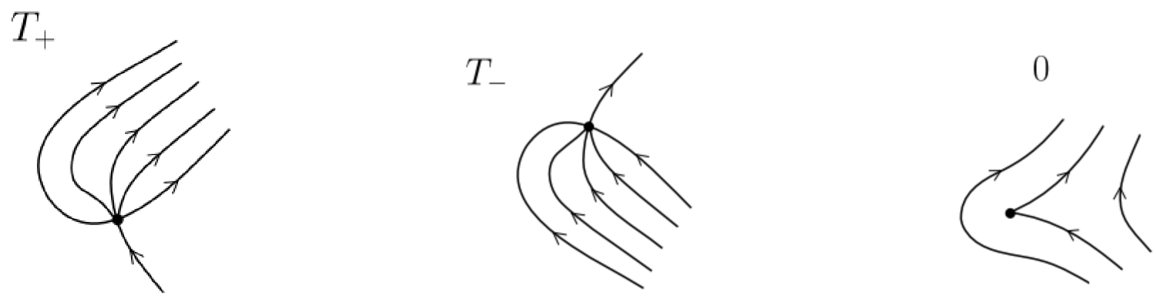


Figure 2: Complete bifurcation diagram of system (1).

3.1 Bifurcation on curves

The $T_\pm$ curve is the only curve we will look at using the β coordinates instead of the γ coordinates. We do this because $||\gamma|| = 0$ for this entire curve. We see that, on the $T_\pm$ curve, the parabolic equilibrium is $(x, y) = (-\frac{\beta_2}{2}, 0)$. This gives us eigenvalues of $\lambda_1 = 0$ and $\lambda_2 = \frac{\beta_2}{2}$ and thus a Fold Bifurcation. This means that for T_+, there is only one stable manifold approaching from below, and multiple unstable manifolds emerging from the equilibrium on the other side. For T_- we have the same but flipped. We can see these portraits in figure 3a and 3b.



(a) Stable and unstable manifolds for the $(0, 0)$ equilibrium with β on the T_+ curve.

(b) Stable and unstable manifolds for the $(0, 0)$ equilibrium with β on the T_- curve.

(c) Stable and unstable manifold for the $(0, 0)$ equilibrium with $||\beta|| = 0$.

Figure 3: Phase portraits of system (1) with β on the $T_\pm$ curves and $||\beta|| = 0$. Originally published in [3].

When $\beta_2 = 0$ we have $(\beta_1, \beta_2) = (0, 0)$. Then we have a Bogdanov-Takens Bifurcation, which is what the Bogdanov normal form unfolds. Conceptually, we see that the two phase portraits of the Fold bifurcation have merged into one, leaving only one stable and one unstable manifold. This means we only see a stable manifold from below and above, as shown in figure 3c.

3.1.1 Curve H

On the H curve we have that $\gamma_1 = 0$ and $\gamma_2 > 0$. This gives us that $\dot{v} = u(u-1) - \gamma_2 uv$. We can see that for $(u, v) = (0, 0)$ we get $\lambda_{1,2} = \pm i$. So we know a Hopf bifurcation will occur. Now we want to know if this is a supercritical or subcritical Hopf bifurcation. We can do this by determining the sign of l_1. The formula for calculating l_1 is given by

$$l_1 = \frac{1}{2\omega^2}\Re[i\langle p, B(q,q)\rangle\langle p, B(q,\bar{q})\rangle + \omega\langle p, C(q,q,\bar{q})\rangle], \tag{35}$$

with $q, p \in \mathbb{C}^2$ and $\omega \in \mathbb{R}$ is defined by $\lambda_{1,2} = \pm i\omega$. We define B and C to be the quadratic and cubic terms of the taylor expansion $X(s) = \mathcal{A}s + \frac{1}{2}B(s,s) + \frac{1}{6}C(s,s,s) + \mathcal{O}(s^4)$, where $X(s)$ is system (3). For (35) to work we have some restrictions on the vectors q and p, namely $\mathcal{A}q = i\omega q$, $\mathcal{A}^T p = -i\omega p$ and $\langle p, q\rangle = \bar{p}^T q = 1$ To keep our calculations neat, we introduce some short-hand notation: $S_1 = \langle p, B(q,q)\rangle$, $S_2 = \langle p, B(q,\bar{q})\rangle$, and $S_3 = \langle p, C(q,q,\bar{q})\rangle$. We can see that $\omega = 1$ and that $S_3 = 0$ as we have no cubic terms in our system. From the Taylor expansion of our system, we see that

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B_1(q,p) = 0 \text{ and } B_2(q,p) = 2q_1p_1 - \gamma_2(q_1p_2 + q_2p_1). \tag{36}$$

Now we can calculate the restrictions on q and p. From $\mathcal{A}q = iq$, we see that $q_2 = iq_1$. Plugging this into $-q_1 = iq_2$, we see that $q_1 = a \in \mathbb{C}$ can be freely chosen and thus $q = a\begin{pmatrix} 1 \\ i \end{pmatrix}$. From $\mathcal{A}^T p = -ip$, we see that $p_2 = ip_1$. Plugging this into $p_1 = -ip_2$, we see that $p_1 = b \in \mathbb{C}$ can be freely chosen and thus $p = b\begin{pmatrix} 1 \\ i \end{pmatrix}$. Now we can see how a and b relate by calculating $\langle p, q\rangle = 1$. We have

$$\begin{aligned} 1 &= \bar{p}_1q_1 + \bar{p}_2q_1 \\ &= a\bar{b} + (-i\bar{b})(ia) \\ &= 2a\bar{b} \\ \bar{b} &= \frac{1}{2a}. \end{aligned} \tag{37}$$

We will do one more thing before starting to calculate l_1. We will WOLG normalize q with $\langle q, q\rangle = 1$. This gives us $a\bar{a} = |a|^2 = \frac{1}{2}$. Now we will calculate S_1, we have

$$\begin{aligned} S_1 &= \bar{p}_1 B_1(q,q) + \bar{p}_2 B_2(q,q) \\ &= -\frac{i}{2a}(2q_1^2 - 2\gamma_2 q_1 q_2) \\ &= -\frac{i}{2a}(2a^2 - 2i\gamma_2 a^2) \\ &= -ia - \gamma_2 a. \end{aligned} \tag{38}$$

For S_2 we get

$$\begin{aligned} S_2 &= \bar{p}_1 B_1(q,\bar{q}) + \bar{p}_2 B_2(q,\bar{q}) \\ &= -\frac{i}{2a}[2q_1\bar{q}_1 - \gamma(q_1\bar{q}_2 + \bar{q}_1 q_2)] \\ &= -\frac{i}{2a}[2a\bar{a} - \gamma_2(a(-i\bar{a}) + ia\bar{a})] \\ &= -i\bar{a}. \end{aligned} \tag{39}$$

Substituting these values into (35), we get

$$\begin{aligned} l_1 &= \frac{1}{2}\Re[iS_1S_2] \\ &= \frac{1}{2}\Re[i(-ia - \gamma_2 a)(-i\bar{a})] \\ &= \frac{1}{2}\Re[-i|a|^2 - \gamma|a|^2] \\ &= -\frac{\gamma_2|a|^2}{2} \\ &= -\frac{\gamma_2}{4}. \end{aligned} \tag{40}$$

Thus, we see that $l_1 = -\frac{\gamma_2}{4}$. As $\gamma_2 > 0$ we see that $l_1 < 0$. So, we have a supercritical Hopf bifurcation. So when our parameters are moving from region 2 to 3 in figure 2, a stable cycle appears when crossing the H curve and the stable equilibrium becomes unstable. We saw that at the H curve, our limit cycle collapsed to a point. Thus, we have a phase portrait that looks like the portrait labeled H in figure 4.

3.1.2 Curve P

On the P curve we have $\gamma_1 = -\frac{1}{7}\gamma_2$ and $\gamma_2 > 0$. This gives us $\dot{v} = u(u-1) + \gamma_2(\frac{1}{7}v - uv)$. At $(0,0)$ we have eigenvalues $\lambda_{1,2} = \frac{\gamma_2}{14} \pm \sqrt{\frac{\gamma_2^2}{196} - 1}$. For $\gamma_2 < 14$ we see that $\frac{\gamma_2^2}{196} - 1 < 0$. As we are looking close to $||\gamma|| = 0$, we have an unstable focus. We have seen, in our exploration of the unique limit cycle, that on the P curve we have a stable homoclinic orbit for $\mathcal{H}(u,v) = \frac{1}{6}$. Thus, our phase portrait looks like the level curve of $\mathcal{H}(u,v)$ at $h = \frac{1}{6}$ in figure 1 with an unstable focus in the center, as displayed in figure 4 as phase portrait P.

3.2 Phase portrait in regions

Now that we have characterized all the curves in our diagram, we can discuss what happens in the regions between these curves. The easiest region to draw a phase portrait for is region 1. Here we have no equilibria, and we see that $\dot{y}$ mostly points up. This gives us the phase portrait displayed in region ① in figure 4.

Region ③ is the one we discussed during our exploration of the unique limit cycle. This gives us a unique cycle and an unstable focus, as seen in figure 4. In region ②, the limit cycle has vanished when crossing the H curve and is transitioning into the T_- curve. This gives us a phase portrait just like we found on the H curve. The only region we have left is region④. This region exhibits the same behaviour as in region ②, only now the limit cycle did not vanish into a point, but broke at the saddle point in the opposite direction of the phase portrait in region ②. This gives us the phase portrait displayed in figure 4. So figure 4 gives us the full bifurcation diagram plus all phase portraits.

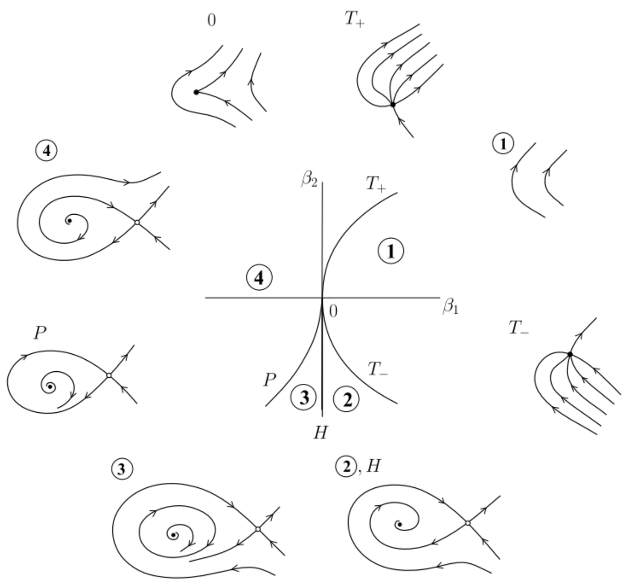


Figure 4: All phase portraits of our bifurcation diagram and its regions. Originally published in [3].

4 Conclusion

So, in conclusion, we have unfurled the phase portraits of the full bifurcation diagram from the Bogdanov normal form. We have also proven that region ③ between the H and P curves has unique limit cycles. If you want to see a more in-depth proof I suggest reading paper [1] by Maoan Han and collaborators and section 8.4 from [3] by Yuri Kuznetsov.

References

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