# Analyse in meer variabelen

51. Let $(V,+,0,\cdot)$ be a vector space of finite dimension $\dim V = n$. Show that an associative unital algebra $(\Lambda,+,0,\cdot,\wedge,1)$ together with a linear mapping $\varphi : V \longrightarrow \Lambda$ are uniquely determined — up to canonical isomorphy — by the two properties

   (i) $\bigwedge_{v\in V} \quad \varphi(v)\wedge\varphi(v) \;=\; 0 \qquad (\text{so } \varphi(w)\wedge\varphi(v) = -\varphi(v)\wedge\varphi(w))$

   (ii) for every linear mapping $h : V \longrightarrow A$ into an associative unital algebra $(A,+,0,\cdot,*,1)$ satisfying
   $$\bigwedge_{v\in V} \quad h(v) * h(v) \;=\; 0$$
   there is a unique 1–preserving algebra-homomorphism $\hat{h} : \Lambda \longrightarrow A$ with $\hat{h}\circ\varphi = h$.

52. Denote by $\mathcal{P} = \mathcal{P}(\{1,\ldots,n\})$ the set of all subsets of $\{1,\ldots,n\}$ and by $\{e_T \mid T\in\mathcal{P}\}$ a fixed basis of $\mathbb{R}^{2^n}$. For $r,s\in\{1,\ldots,n\}$ put
   $$\sigma(r,s) \;:=\; \begin{cases} 1 & r<s \\ 0 \quad \text{if} & r=s \\ -1 & r>s \end{cases}$$
   and
   $$\tau(R,S) \;:=\; \prod_{r\in R}\prod_{s\in S}\sigma(r,s)$$
   for $R,S\in\mathcal{P}$. Show that
   $$e_R \;\wedge\; e_S \;:=\; \tau(R,S)\cdot e_{R\cup S}$$
   turns $\mathbb{R}^{2^n}$ into an associative unital algebra $\Lambda(e_T\,|\,T\in\mathcal{P})$ with $e_\emptyset = 1$. Check that
   $$\bigwedge_{i,j\in\{1,\ldots,n\}} \quad e_{\{j\}} \;\wedge\; e_{\{i\}} \;=\; -e_{\{i\}} \;\wedge\; e_{\{j\}}$$
   and that
   $$e_T \;=\; e_{\{i_1\}} \;\wedge\; e_{\{i_2\}} \;\wedge\; \ldots \;\wedge\; e_{\{i_k\}}$$
   for all $T = \{i_1,\ldots,i_k\}$ with $i_1 < \ldots < i_k$. Let $V$ be a vector space with basis $\{e_1,\ldots,e_n\}$. Prove that $\Lambda := \Lambda(e_T\,|\,T\in\mathcal{P})$ together with $\varphi : V \longrightarrow \Lambda$ defined by $\varphi(e_i) = e_{\{i\}}$, $i = 1,\ldots,n$ satisfy $(i)$ and $(ii)$ of the previous exercise.

   This unique algebra $\Lambda = \Lambda(V)$ is called the *exterior (or Graßmann) algebra* of $V$. Generalize from $\mathbb{R}$ to any field $K$. Can you also generalize to a ring $R$ ? How important is the assumption of finite dimension?

53. Let $\Lambda$ be the exterior algebra of an $n$–dimensional vector space $V$ with basis $\{e_1, \ldots, e_n\}$.

$(i)$ Explain that the $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $i_1 < \ldots < i_k$ together with 1 form a basis of $\Lambda$.

$(ii)$ Check that for all subsets $\{i_1 < \ldots < i_k\}, \{j_1 < \ldots < j_\ell\} \subseteq \{1, \ldots, n\}$ one has

$$e_{j_1} \wedge \ldots \wedge e_{j_\ell} \ \wedge \ e_{i_1} \wedge \ldots \wedge e_{i_k} \quad = \quad (-1)^{kl} e_{i_1} \wedge \ldots \wedge e_{i_k} \ \wedge \ e_{j_1} \wedge \ldots \wedge e_{j_\ell} \ .$$

$(iii)$ Show that $\{v_1, \ldots, v_m\} \subseteq V$ is linear dependent if and only if $v_1 \wedge \ldots \wedge v_m = 0$.

$(iv)$ Let $\{u_1, \ldots, u_m\}, \{v_1, \ldots, v_m\} \subseteq V$ with $\{u_1, \ldots, u_m\}$ linear independent. Prove that $\langle u_1, \ldots, u_m \rangle = \langle v_1, \ldots, v_m \rangle$ have the same linear span if and only if $u_1 \wedge \ldots \wedge u_m$ is a scalar multiple of $v_1 \wedge \ldots \wedge v_m$.

Define $\Lambda^k V := \langle v_1 \wedge \ldots \wedge v_k \mid v_i \in V \rangle$, the $k$th exterior power of $V$.

$(v)$ Show that $\Lambda^k V = 0$ for $k > n$ and that for $k \leq n$ the $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $i_1 < \ldots < i_k$ with the now fixed index $k$ form a basis of $\Lambda^k V$. Conclude $\dim \Lambda^k V = \binom{n}{k}$. Identify $V = \Lambda^1 V$ and check

$$\Lambda \quad = \quad \bigoplus_{k=0}^{n} \Lambda^k V$$

and

$$\bigwedge_{k,\ell \in \mathbb{N}_0} \quad \Lambda^k V \ \wedge \ \Lambda^\ell V \quad = \quad \Lambda^{k+\ell} V$$

where $A \wedge B := \langle a \wedge b \mid a \in A, b \in B \rangle$.

$(vi)$ For $\varphi : V \longrightarrow \Lambda(V)$ and $\psi : W \longrightarrow \Lambda(W)$ show that for every $f \in L(V, W)$ there is a unique 1–preserving algebra-homomorphism $\Lambda(f) : \Lambda(V) \longrightarrow \Lambda(W)$ satisfying $\Lambda(f) \circ \varphi = \psi \circ f$.

$(vii)$ Prove that $\Lambda(f \circ g) = \Lambda(f) \circ \Lambda(g)$ if $g \in L(U, V)$ for some other vector space $U$ with exterior algebra $\Lambda(U)$.

$(viii)$ Check that $\Lambda(f)(\Lambda^k V) \subseteq \Lambda^k W$, yielding $\Lambda^k f \in L(\Lambda^k V, \Lambda^k W)$ and the splitting $\Lambda(f) = \bigoplus \Lambda^k(f)$. How is $\Lambda^n(f)$ for $f \in L(V)$ related to $\det(f)$ ?

$(ix)$ Explain why $v_1 \wedge \ldots \wedge v_m$ can be interpreted as the $m$–dimensional parallelepiped with sides $v_1, \ldots, v_m$.

$(x)$ Let $V^* = L(V, \mathbb{R})$ be the dual space of $V$. Validate

$$\bigoplus_{k=0}^{n} (\Lambda^k V)^* \quad = \quad (\Lambda V)^* \quad = \quad \Lambda(V^*) \quad = \quad \bigoplus_{k=0}^{n} \Lambda^k V^*$$

and prove that

$$(\alpha_1 \wedge \ldots \wedge \alpha_k)(v_1, \ldots, v_k) \ := \ (\alpha_1 \wedge \ldots \wedge \alpha_k)(v_1 \wedge \ldots \wedge v_k) \ = \ \det\left(\alpha_i(v_j)\right)_{i,j=1,\ldots,k}$$

for all $\alpha_1, \ldots, \alpha_k \in V^*$ and $v_1, \ldots, v_k \in V$.