1.1 Phase portraits

Rewrite the second order equation $\ddot{x} = -\frac{\partial V}{\partial x}$ as a first order system with energy function $E : \mathbb{R}^2 \to \mathbb{R}$, $E(x, y) = \frac{1}{2}y^2 + V(x)$. For several shapes of the graph of $V$, sketch the phase portrait of the vector field $(\dot{x}, \dot{y})$. Recall that the integral curves are given by the level sets of the function $E$. What do they look like in the cases where $V$ has a local maximum or minimum or a horizontal asymptote? How do the integral curves intersect the $x$–axis?

1.2 Gradient- and Hamiltonian vector fields

Let $E : \mathbb{R}^2 \to \mathbb{R}$ be a smooth energy function. Consider the (Hamiltonian) vector field $X = (\partial E/\partial y) e_1 - (\partial E/\partial x) e_2$ and the gradient vector field $\text{grad} E = (\partial E/\partial x) e_1 + (\partial E/\partial y) e_2$. Prove that at each point of $\mathbb{R}^2$, the integral curves of $X$ and $\text{grad} E$ are orthogonal. What is the relation to the level curves of $E$? Discuss the changes in the phase portraits of $X$ and $\text{grad} E$ if $E$ is replaced by $-E$. In particular consider a neighbourhood of a minimum and a saddle point of $E$.

1.3 Lissajous figures

Consider the two-degree-of-freedom system consisting of a particle of unit mass moving in the plane with potential energy $U(x_1, x_2) = \frac{1}{2}(x_1^2 + \omega^2 x_2^2)$. Show that the general solution is of the form $(x_1(t), x_2(t)) = (A_1 \sin(t + \varphi_1), A_2 \sin(\omega t + \varphi_2))$. Discuss the orbits in the $(x_1, x_2)$ plane for the cases where $\omega$ is 1 or 2 and the phase difference, $\varphi_1 - \varphi_2$, has the value 0, $\pi/2$ or $\pi$. Show that for integer $\omega$ there is a phase difference and amplitudes $A_1$ and $A_2$ such that the orbit in the $(x_1, x_2)$ plane agrees with the graph of a Chebychev polynomial. Show that for rational $\omega$ the orbits are closed, and for irrational $\omega$ the orbits densely fill a rectangle in the $(x_1, x_2)$ plane.

1.4 Period and area

Let $E : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function given by $E(x, y) = \frac{1}{2}y^2 + V(x)$ and assume that for $E_0 \in \mathbb{R}$, the motion in the level set $E^{-1}(E_0)$ is periodic. Show that for energies near $E_0$ the motion is also periodic. Let $A(E_0)$ denote the area enclosed by the level set $E^{-1}(E_0)$, and let $T(E_0)$ denote the period of the motion in this level set. Show that $T(E_0) = \left. \frac{dA(z)}{dz} \right|_{z=E_0}$. 