Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 28 September.

3.1 Asymptotics of the period of the pendulum

Consider the Hamiltonian function $H(x,y) = \frac{1}{2}y^2 - \omega^2 \cos x$ of the pendulum. For $|z| < \omega^2$ consider the level set $H^{-1}(z)$. What is the amplitude of oscillation in this level? If $T(z)$ denotes the period of oscillation in this level, then give an explicit integral expression for this. Determine $\lim\limits_{z \to -\omega^2} T(z)$ and $\lim\limits_{z \to \omega^2} T(z)$.

3.2 Rotating pendulum

Analyse the dynamics of the rotating pendulum $\ddot{x} = M - \sin x$ in dependence of $M \in \mathbb{R}$. Identify the values of the parameter M where the dynamical behaviour of the system changes and give for each of the resulting open regions in parameter space at least one phase portrait. Is it possible to write the system as a Lagrangean system on the cyinder $S^1 \times \mathbb{R}$?

3.3 The spherical pendulum

A spherical pendulum has length ℓ and mass m. Let g be the acceleration of gravity.

1. Derive the equations of motion from the variational principle.
2. Determine two (first) integrals, or conserved quantities.
3. Give the Hamiltonian equations for the system, in which the conservation laws are well expressed. Reduce to one degree of freedom (as in the central force field problem).
4. Describe the dynamics of the spherical pendulum in terms of this reduction. First describe the geometry of the invariant level sets defined by the conserved quantities and second characterise the corresponding dynamics. Interprete these findings in the configuration space. Why is this description not complete?

3.4 A bead on a wire, Huygens's isochronous curve

A bead with unit mass moves along a stiff wire, without friction. The wire lies in a vertical plane, the acceleration of gravity equals 1. Suppose the wire is given by the equation $y = U(x)$. Show that the system has energy $H = [1 + (\frac{\mathrm{d}U}{\mathrm{d}x})^2] \cdot \frac{\dot{x}^2}{2} + U(x)$. Let q be an arclength parameter along the wire and put $p := \dot{q}$. Show that the system has the Hamiltonian form $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$. Next assume that U attains a minimum at $x = x_0$. Prove that the frequency of 'small oscillations' at $x = x_0$ is equal to $\sqrt{U''(x_0)}$.

Subsequently we consider the special case where U is a cycloid, parametrically given by $\vartheta \mapsto (x,y) = (a(2\vartheta + \sin 2\vartheta), a(1 - \cos 2\vartheta))$. Here $a > 0$ is a constant while ϑ varies over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Compute its arclength parameter and obtain its equations of motion. Prove that this device implements the harmonic oscillator.