7.1 Poisson and Lie brackets

In this exercise we study some properties of Poisson and Lie brackets on symplectic manifolds.

1. Show that if two functions $F, G$ are integrals of $H$ then so is $\{F, G\}$.

2. Show that if $\phi : \mathcal{P} \to \mathcal{P}$ is symplectic then $\{F, G\} \circ \phi = \{F \circ \phi, G \circ \phi\}$. What is the meaning of this equation?

3. Show that the map $F \mapsto \mathbf{X}_F$ is a Lie algebra anti-homomorphism between the Lie algebras $(C^\infty(\mathcal{P}), \{,\})$ of smooth functions $\mathcal{P} \to \mathbb{R}$ and the Lie algebra $(\mathcal{F}^\infty(\mathcal{P}), [,])$ of smooth Hamiltonian vector fields on $\mathcal{P}$. In other words, show that $[\mathbf{X}_F, \mathbf{X}_G] = -\mathbf{X}_{\{F, G\}}$ where the Lie bracket $[[X, Y]]$ of vector fields $X, Y$ is the vector field defined by $[[X, Y]](F) = X(Y(F)) - Y(X(F))$ for any function $F : \mathcal{P} \to \mathbb{R}$.

7.2 The orthogonal group $O(n, \mathbb{R})$

Let $\mathfrak{gl}(n, \mathbb{R})$ be the set of all real $n \times n$-matrices. Further define

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\begin{align*}
\text{Gl}(n, \mathbb{R}) & = \{ S \in \mathfrak{gl}(n, \mathbb{R}) \mid \det S \neq 0 \} \\
o(n, \mathbb{R}) & = \{ S \in \mathfrak{gl}(n, \mathbb{R}) \mid S^T S = \text{id} \} \\
o(n, \mathbb{R}) & = \{ A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^T = -A \} \\
\text{Sym}(n, \mathbb{R}) & = \{ A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^T = A \}.
\end{align*}
\]

1. Show that $o(n, \mathbb{R})$ and $\text{Sym}(n, \mathbb{R})$ are real vector spaces. Give their dimensions.

2. Show that $\mathfrak{gl}(n, \mathbb{R})$ and $o(n, \mathbb{R})$ form Lie algebras with respect to the product given by the commutator.

3. Show that $\text{Gl}(n, \mathbb{R})$ is an $n^2$-dimensional manifold. Show how $\mathfrak{gl}(n, \mathbb{R})$ can be regarded as the tangent space $T_{\text{id}}\text{Gl}(n, \mathbb{R})$.

4. Let $F : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ be defined by $F(S) = S^T S$. Show that $F$ is a smooth map and that the image of $F$ is a subset of $\text{Sym}(n, \mathbb{R})$.

5. Show that the derivative$^1$ $D_{\text{id}}F : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ is given by $D_{\text{id}}F(B) = B^T + B$. What is the rank of this derivative?

6. Show that the rank of the derivative $D_SF : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ is independent of $S \in O(n, \mathbb{R})$. Hint: Use the fact that $O(n, \mathbb{R})$ is a group.

7. Show that $O(n, \mathbb{R})$ is a manifold and determine its dimension. In what sense can $o(m, \mathbb{R})$ be regarded as the tangent space $T_{\text{id}}O(n, \mathbb{R})$?

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$^1$In another notation, $D_{\text{id}}F = F_* \text{id}$. 