Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 15 November.

10.1 Motion of a charge in an electromagnetic field

Consider a particle of mass $m$ and charge $e$ moving in $\mathbb{R}^3$ under the influence of a magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ and an electric field $\mathbf{E} = (E_x, E_y, E_z)$. The electric field is conservative and can thus be written as $\mathbf{E} = -\nabla \phi$ where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is called the electric potential. From physics we know that Newton’s equations of motion for the particle are

$$m \ddot{\mathbf{r}} = e \dot{\mathbf{r}} \times \mathbf{B} + e \mathbf{E}.$$ 

Define the symplectic 2-form

$$\omega = dp_x \wedge dx + dp_y \wedge dy + dp_z \wedge dz + e (B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy)$$

and the Hamilton function $H(p, q) = \frac{p^2}{2m} + D_e (e^{-2a} - 2e^{-aq})$, where $D_e$ and $a$ are positive constants, and $(p, q) \in \mathbb{R}^2$.

1. A Morse oscillator is often used to describe a chemical bond. What is the meaning of $D_e$ in this case?

2. Show that there are two critical energies, $E_1 < E_2$, such that the level sets $M_E = \{(p, q) \in \mathbb{R}^2 : H(p, q) = E\}$ are empty if $E < E_1$, topological circles if $E_1 < E < E_2$, and topological lines for $E > E_2$. Plot the level sets for the energies $E_1$, $E_2$, an energy between $E_1$ and $E_2$, and an energy greater than $E_2$. 

10.2 The Morse oscillator

Consider the Morse oscillator described by the Hamilton function

$$H = \frac{p^2}{2m} + D_e (e^{-2a} - 2e^{-aq}) ,$$

where $D_e$ and $a$ are positive constants, and $(p, q) \in \mathbb{R}^2$.

1. A Morse oscillator is often used to describe a chemical bond. What is the meaning of $D_e$ in this case?

2. Show that there are two critical energies, $E_1 < E_2$, such that the level sets

$$M_E = \{(p, q) \in \mathbb{R}^2 : H(p, q) = E\}$$

are empty if $E < E_1$, topological circles if $E_1 < E < E_2$, and topological lines for $E > E_2$. Plot the level sets for the energies $E_1$, $E_2$, an energy between $E_1$ and $E_2$, and an energy greater than $E_2$. 
3. For $E_1 < E < E_2$, compute the area, $A(E)$, of the region enclosed by the level set $M_E$ in the $(p,q)$ plane. For such energies, compute the period

$$T(E) = \frac{dA(E)}{dE}.$$ 

Sketch and interpret the graph of the period $T(E)$ for energies $E_1 < E < E_2$.

### 10.3 Reduced Euler top

Analyse the dynamics of the “reduced Euler top”

$$H(x,y,z) = \frac{x^2}{2a} + \frac{y^2}{2b} + \frac{z^2}{2c}, \quad 0 < a \leq b \leq c$$
on $S^2$ in the limiting cases $a \to b$ and $b \to c$.

### 10.4 Another Hamiltonian system on $S^2$

In this exercise we study another Hamiltonian system defined on $S^2$ which is not a cotangent bundle.

In $\mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$ consider the submanifold $S^2 = \{x \in \mathbb{R}^3 : x^2 = 1\}$ and the 2-form

$$\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2.$$

1. Show that $\omega$ is not closed. Show that $\omega$ is degenerate in $\mathbb{R}^3$ by finding at each point $x \in \mathbb{R}^3$ the space $N_x = \{\xi \in T_x \mathbb{R}^3 : \omega(\xi, -) = 0\}$. Show that the restriction $\varpi = \omega|_{S^2}$ of $\omega$ to $S^2$ is closed and non-degenerate. Show that $\varpi$ is not exact.

2. Let $H : S^2 \to \mathbb{R}$. Find the Hamiltonian vector field $X_H$ on $S^2$ that satisfies $\varpi(X_H, -) = dH(-)$.

3. Compute the Poisson brackets $\{x_i, x_j\}, \ i,j = 1,2,3$ with respect to $\varpi$ and then compute the Poisson bracket $\{F,G\}$ for two arbitrary functions $F,G : S^2 \to \mathbb{R}$.

4. Describe the dynamics of the Hamiltonian function $H = x_1$ on $S^2$.

5. Show that every locally Hamiltonian vector field $X$ on $S^2$ is globally Hamiltonian, given that every boundaryless 1-chain on $S^2$ is the boundary of a 2-chain.

*Hint.* Use appropriate coordinates on $S^2$ (but be careful!).