The homework consists of two exercises, to be handed in on Tuesday 23 November.

11.1 Revisiting constant gravity

Consider the motion of a particle under a gravitational force described by the Hamiltonian function

\[ H(x, y) = \frac{y_1^2 + y_2^2}{2m} + mgx_2 \]

on the phase space \( \mathbb{R}^4 \) with co-ordinates \((x_1, x_2, y_1, y_2)\). Find a complete solution of the Hamilton–Jacobi equation for this problem.

11.2 Revisiting the Kepler problem

The Hamilton–Jacobi equation for the Kepler problem is separable in polar co-ordinates. Furthermore, because of the degeneracy of the problem (all orbits are periodic which implies the existence of “too many” integrals), the Hamilton–Jacobi equation is also separable in other co-ordinates.

Consider the Kepler Hamiltonian on \( T^*\mathbb{R}^2 \) given by

\[ H(x, y) = \frac{y_1^2 + y_2^2}{2} - \frac{1}{\sqrt{x_1^2 + x_2^2}}. \]

1. Express \( H \) in terms of the co-ordinates

\[ q_1 = \sqrt{x_1^2 + x_2^2} + x_1 \]
\[ q_2 = \sqrt{x_1^2 + x_2^2} - x_1 \]

and their conjugate momenta \( p_1, p_2 \).

2. Use the Hamilton–Jacobi method in order to obtain a generating function such that in the new co-ordinates three integrals of the problem become manifest (i.e., three of the new co-ordinates are constant in time). Give a closed expression for the generating function.

3. Express the integrals obtained from the Hamilton–Jacobi method in Cartesian and polar co-ordinates.

Hint: Choose wisely the type of the generating function for the Hamilton–Jacobi method.