Conservative Dynamical Systems 2010/2011

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The homework consists of two exercises, to be handed in on Tuesday 23 November.

11.1 Revisiting constant gravity

Consider the motion of a particle under a gravitational force described by the Hamiltonian function

$$H(x,y) \;=\; \frac{y_1^2+y_2^2}{2m} \;+\; mgx_2$$

on the phase space $\mathbb{R}^4$ with co-ordinates (x_1,x_2,y_1,y_2). Find a complete solution of the Hamilton–Jacobi equation for this problem.

11.2 Revisiting the Kepler problem

The Hamilton–Jacobi equation for the Kepler problem is separable in polar co-ordinates. Furthermore, because of the degeneracy of the problem (all orbits are periodic which implies the existence of "too many" integrals), the Hamilton–Jacobi equation is also separable in other co-ordinates.

Consider the Kepler Hamiltonian on $T^*\mathbb{R}^2$ given by

$$H(x,y) \;=\; \frac{y_1^2+y_2^2}{2} \;-\; \frac{1}{\sqrt{x_1^2+x_2^2}} \;.$$

1. Express H in terms of the co-ordinates

$$\begin{aligned} q_1 &= \sqrt{x_1^2+x_2^2} \;+\; x_1 \\ q_2 &= \sqrt{x_1^2+x_2^2} \;-\; x_1 \end{aligned}$$

   and their conjugate momenta p_1, p_2.

2. Use the Hamilton–Jacobi method in order to obtain a generating function such that in the new co-ordinates three integrals of the problem become manifest (i.e., three of the new co-ordinates are constant in time). Give a closed expression for the generating function.

3. Express the integrals obtained from the Hamilton–Jacobi method in Cartesian and polar co-ordinates.

Hint: Choose wisely the type of the generating function for the Hamilton–Jacobi method.