Conservative Dynamical Systems 2010/2011

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The homework consists of two exercises, to be handed in on Tuesday 7 December.

13.1 Lemma of Paley–Wiener

Prove the lemma of Paley–Wiener: a periodic function with Fourier series

$$f(x) \;=\; \sum_{k\in\mathbb{Z}} f_k \,\mathrm{e}^{2\pi \mathrm{i} kx}$$

is (real) analytic if and only if the coefficients decay exponentially fast:

$$\bigvee_{M,\eta>0} \;\; \bigwedge_{k\in\mathbb{Z}} \;\; |f_k| \;\le\; M\cdot \mathrm{e}^{-|k|\cdot\eta} \;\; .$$

13.2 A Poincaré–Birkhoff fixed point theorem

Consider the annulus $A := \mathbb{T}^1 \times [1,2]$, with coordinates (x,y), where x is counted mod 1. Consider a smooth, boundary preserving diffeomorphism $T_\varepsilon : A \longrightarrow A$ of the form $T_\varepsilon : (x,y) \mapsto (x+\rho(y), y) + \varepsilon(f(x,y,\varepsilon), g(x,y,\varepsilon))$ and such that

- $\rho'(y) \neq 0$, saying that T_ε is a twist-map (for simplicitiy we take ρ increasing);
- $\displaystyle\oint_\gamma y\,\mathrm{d}x = \oint_{T_\varepsilon(\gamma)} y\,\mathrm{d}x$, which means that T_ε is preserves area.

Show that for each rational number $\dfrac{p}{q}$, with

$$\rho(1) \;<\; \frac{p}{q} \;<\; \rho(2) \;\; ,$$

in A there exists a periodic point of T_ε, of period q, provided that $|\varepsilon|$ is sufficiently small. (Hint: Abbreviating $T^q_\varepsilon(x,y) = (\Phi_{q,\varepsilon}(x,y), y + O(\varepsilon))$, with $\Phi_{q,\varepsilon}(x,y) = x + q\rho(y) + O(\varepsilon)$, consider the equation $\Phi_{q,\varepsilon}(x,y) = x + p$, for $p \in \mathbb{Z}$. Use the implicit function theorem in order to obtain a curve $C = \{y = F(x,\varepsilon)\}$ of solutions. Then study the intersection of C and $T^q_\varepsilon(C)$.)