Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 14 December.

14.1 Diophantine numbers: A thick Cantor set

In the unit interval $[0,1]$, for given constants $\gamma > 0$ and $\tau > 2$, consider a subset $D_{\gamma,\tau}$ of Diophantine numbers, defined as follows. We say that $\rho \in D_{\gamma,\tau}$ if for all rational numbers p/q one has

$$\left| \rho - \frac{p}{q} \right| \quad \geq \quad \frac{\gamma}{q^\tau} \ .$$

Show that $D_{\gamma,\tau}$ is nowhere dense. Also show that the Lebesgue measure of $[0,1] \setminus D_{\gamma,\tau}$ is of order $O(\gamma)$ as $\gamma \to 0$.

14.2 A small divisor problem by Sternberg

On $\mathbb{T}^2$, with coordinates (x_1, x_2), a vector field X is given, with the following property. If C_1 denotes the circle $C_1 := \{x_1 = 0\}$, then the Poincaré return mapping $P : C_1 \to C_1$ with respect to X is a rigid rotation $x_2 \mapsto P(x_2) = x_2 + \rho$, everything counted mod 1. From now on we abbreviate $x := x_2$. Let $f(x)$ be the return time of the integral curve connecting the points x and $P(x)$ in C_1. A priori, f does not have to be constant. The problem now is to construct a(nother) circle C_2, that does have a constant return time. To this purpose let φ_t denote the flow of X and express P in terms of φ_t and f. Let us look for a circle C_2 of the form

$$C_2 \quad = \quad \left\{ \ \varphi_{\alpha(x)}(x) \ \mid \ x \in C_1 \ \right\} \ .$$

So the search is for a (periodic) function α and a constant c, such that

$$\varphi_c(C_2) \quad = \quad C_2 \ .$$

Rewrite this equation explicitly in terms of α and c. Solve this equation formally in terms of Fourier series. What condition on ρ in general will be needed? Give conditions on ρ, such that for a real analytic function f a real analytic solution α exists.

14.3 The flux form

Let $\mathcal{P}$ be a $(2n)$–dimensional symplectic manifold with symplectic 2–form ω, and $c \in \mathbb{R}$ be a regular value of the Hamiltonian function $H : \mathcal{P} \longrightarrow \mathbb{R}$. Consider on the energy surface $E_c = H^{-1}(c)$ the volume form Ω_c, see exercise 8.3, and show that

$$\iota_{X_H} \Omega_c \quad = \quad \frac{n!}{(n-1)!} \omega \wedge \ldots \wedge \omega$$

(the $(n-1)$–fold wedge product). This is the flux form of transition state theory.