Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 17 February.

2.1 The Legendre transformation

Let $H(x,p) = \dfrac{p^2}{2m} + V(x)$ be the energy function on $\mathbb{R}^2$, where $V : \mathbb{R} \longrightarrow \mathbb{R}$ is the potential energy. Compute the Legendre transformation

$$L(x,v) \quad := \quad \sup_{p \in \mathbb{R}} \left(v \cdot p \; - \; H(x,p) \right)$$

and clarify the situation with a figure. The function L obtained this way is the Lagrangean function of the system. Show that Hamilton's equations $\dot{x} = \dfrac{\partial H}{\partial p}$, $\dot{p} = -\dfrac{\partial H}{\partial x}$ turn into

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial v} \quad = \quad \frac{\partial L}{\partial x} \; .$$

2.2 Asymptotics of the period of the pendulum

Consider the Hamiltonian function $H(x,y) = \frac{1}{2}y^2 - \omega^2 \cos x$ of the pendulum. For $|z| < \omega^2$ consider the level set $H^{-1}(z)$. What is the amplitude of oscillation in this level? If $T(z)$ denotes the period of oscillation in this level, then give an explicit integral expression for this. Determine $\lim\limits_{z \to -\omega^2} T(z)$ and $\lim\limits_{z \to \omega^2} T(z)$.

2.3 Isoperimetric problem

In $\mathbb{R}^3 = \{x, y, z\}$ consider the (x,y)–plane. In the (x,y)–plane a curve is given, connecting the points $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$. Revolve this curve around the x–axis. For which curve does the corresponding surface of revolution have minimal area.

2.4 Rotating pendulum

Analyse the dynamics of the rotating pendulum $\ddot{x} = M - \sin x$ in dependence of $M \in \mathbb{R}$. Identify the values of the parameter M where the dynamical behaviour of the system changes and give for each of the resulting open regions in parameter space at least one phase portrait (drawn by hand or using a computer, at your choice). Is it possible to write the system as a Lagrangean system on the cyinder $S^1 \times \mathbb{R}$?