Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 17 March.

6.1 Steiner ellipse

A particle P of unit mass moves in the plane of a given fixed triangle $A_1A_2A_3$. The force F_i on P is directed towards A_i and is equal to $\Gamma\overline{PA_i}$ for $i = 1, 2, 3$, where Γ is a positive constant, and $\overline{PA_i}$ denotes the distance between P and A_i. Prove that there is a motion of P the path of which coincides with the Steiner ellipse S of $A_1A_2A_3$ (the ellipse S passes through the vertices and the tangent at each vertex is parallel to the opposite side). Show moreover that P covers the three arcs A_1A_2, A_2A_3 and A_3A_1 of S in equal time.

6.2 A harmonic n–body problem

The particles A_i with masses m_i $(i = 1, 2, \ldots, n)$ move in three-dimensional space. Any two distinct points A_i, A_j attract each other by the force $F_{ij} = \Gamma m_i m_j d_{ij}$, where $\Gamma > 0$ and d_{ij} denotes the distance $\overline{A_iA_j}$. We suppose that the motions of A_i and A_j are not disturbed if they pass simultaneously through the same point. Determine the general motion of the particles.

6.3 A special harmonic motion

In Euclidean 3–space, the lines ℓ_i $(i = 1, 2, 3)$ are given, all three passing through the point O. The angle between any of the two lines is α $(0 < \alpha < \pi/2)$. Three particles P_i of unit mass move along the lines ℓ_i, respectively. Any two particles P_i, P_j $(i \neq j)$ attract each other by the force $\Gamma\overline{P_iP_j}$, where Γ is a positive constant and $\overline{P_iP_j}$ denotes the distance between P_i and P_j. Determine the general motion of the three particles.

6.4 Small oscillations

In a plane Π, a fixed homogeneous rod (length $2a$, mass density s, midpoint O) is given. A particle P of mass M moves in Π. It is attracted by any mass element dm at a point Q on the rod by the force $\Gamma M r^\alpha\, dm$. Here, Γ is a positive constant, $\overline{PQ} = r$ and $\alpha = 2n - 1$ for $n = 1, 2, 3, \ldots$. Obviously O is a stable equilibrium point. Determine the frequencies of small oscillations of P in the neighbourhood of O.