Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 7 April.

9.1 The standard map

In this exercise we look briefly at the famous standard map, see B. V. Chirikov, *A universal instability of many-dimensional oscillator systems*, Phys. Rep. **52**: 263 (1979). Many other maps can be locally reduced to the standard map thus the latter describes in a sense the universal behavior of systems with mixed chaotic and ordered motions.

Consider $T^*\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$ with coordinates (q, p) and the symplectic form $\omega = dq \wedge dp$. The *standard map* is given by

$$(Q, P) := \phi(q, p) = (q + p + \varepsilon \sin(q), p + \varepsilon \sin(q)),$$

where $\varepsilon \geq 0$.

1. Show that ϕ is symplectic.
2. Find the fixed points of ϕ and determine their linear stability.
3. Find a generating function $S_2(q, P)$ for ϕ.
4. Given a Hamiltonian H, let $S(q, P, t) = qP - tH(q, P)$. Show that S generates a canonical transformation that is a first order approximation to the flow of H for small t, so it defines a (not very accurate!) symplectic integrator.
5. Deduce that the standard map is the time–1 symplectic integrator of the pendulum.

9.2 Hamilton–Jacobi method

In this exercise we use the Hamilton–Jacobi method in order to solve the equations of motion. Although this problem can be trivially solved by integrating directly the equations of motion, it gives you the opportunity to apply the Hamilton–Jacobi method in a familiar situation.

Consider the motion of a particle with mass $m = 1$ on a vertical plane in a uniform gravitational field with $g = 1$. Assume that the particle is launched at time $t = 0$ from the point (x^0, z^0) with initial momentum (p_x^0, p_z^0).

1. Integrate the equations of motion using the Hamilton–Jacobi method with generating function $S_2(q, P)$.
2. Integrate the equations of motion by modifying appropriately the Hamilton–Jacobi method in order to use a generating function $S_3(p, Q)$.

Hint. Recall that the matrices $\partial^2 S_2(q, P)/\partial q\, \partial P$ and $\partial^2 S_3(p, Q)/\partial p\, \partial Q$ must be invertible.

9.3 Revisiting the Kepler problem

The Hamilton–Jacobi equation for the Kepler problem is separable in polar coordinates. Furthermore, because of the degeneracy of the problem (all orbits are periodic which implies the existence of "too many" integrals), the Hamilton–Jacobi equation is also separable in other coordinates.

Consider the Kepler Hamiltonian on $T^*\mathbb{R}^2$ given by

$$H = \frac{p_x^2 + p_y^2}{2} - \frac{1}{\sqrt{x^2 + y^2}}.$$

1. Express H in terms of the coordinates

$$u = \sqrt{x^2 + y^2} + x,$$
$$v = \sqrt{x^2 + y^2} - x,$$

   and their conjugate momenta p_u, p_v.
2. Use the Hamilton–Jacobi method in order to obtain a generating function such that in the new coordinates three integrals of the problem become manifest (i.e., three of the new coordinates are constant in time). Give a closed expression for the generating function.
3. Express the integrals obtained from the Hamilton–Jacobi method in Cartesian and polar coordinates.

Hint. Choose wisely the type of the generating function for the Hamilton–Jacobi method.