Conservative Dynamical Systems

The following exercise is homework, to be handed in on 28 April.

11.1 The planar isotropic harmonic oscillator revisited

Consider the linear $\mathbb{S}^1$ action on phase space $\mathbb{R}^4$ given by

$$t, \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}.$$

1. Compute the vector field X the flow of which is the above $\mathbb{S}^1$ action.
2. Show that the vector field X is the Hamiltonian vector field of the planar isotropic harmonic oscillator which is described by the Hamiltonian function

   $$J(q,p) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2).$$

   From now on we write $X = X_J$.
3. Introduce complex coordinates $z_k = q_k + ip_k$, $k = 1, 2$ in $\mathbb{R}^4$. Express the above $\mathbb{S}^1$ action in terms of the coordinates z_1, z_2.
4. Define the polynomials $\pi_1 = z_1\bar{z}_1$, $\pi_2 = z_2\bar{z}_2$, $\pi_3 = \mathrm{Re}(z_1\bar{z}_2)$, $\pi_4 = \mathrm{Im}(z_1\bar{z}_2)$. Show that π_k, $k = 1, 2, 3, 4$ are left invariant by the above $\mathbb{S}^1$ action.
5. Give an argument to show that the invariance of π_k with respect to the above $\mathbb{S}^1$ action implies that $\{\pi_k, J\} = 0$ (without computing directly the Poisson brackets).
6. Show that any polynomial that is invariant with respect to the above $\mathbb{S}^1$ action can be expressed through π_k. Hint: first try to determine the form of any arbitrary monomial in such invariant polynomial using complex coordinates.
7. Write $J = (\pi_1 + \pi_2)/2$ and $R = (\pi_1 - \pi_2)/2$, $S = \pi_3$, $T = \pi_4$. Show that $J^2 = R^2 + S^2 + T^2$.

   According to the theory this implies that the reduced phase space $J^{-1}(j)/\mathbb{S}^1$ is diffeomorphic to the set

   $$P_j = \{(R, S, T) \in \mathbb{R}^3 : R^2 + S^2 + T^2 = j^2\}$$

   which is a two-dimensional sphere for $j \neq 0$ and a single point for $j = 0$. The next question asks you to prove part of this fact.

8. Show that there is an one-one and onto map between the set of orbits of X_J with fixed value $J = j$ and the points of P_j.

9. Compute the Poisson brackets between the quantities $(J(q, p), R(q, p), S(q, p), T(q, p))$ in $\mathbb{R}^4$ and express the result in terms of (J, R, S, T). The fact that these Poisson brackets can be expressed through (J, R, S, T) is not coincidental. Can you explain why this happens?

10. Recall that the reduced symplectic form ϖ_j is defined through the relation $\iota_j^*\omega = \rho_j^*\varpi_j$ where ρ_j is the reduction map

$$\rho_j : J^{-1}(j) \to P_j : (q, p) \mapsto (R(q, p), S(q, p), T(q, p)),$$

ι_j is the inclusion map

$$\iota_j : J^{-1}(j) \to \mathbb{R}^4 : (q, p) \mapsto (q, p),$$

and ω is the standard symplectic form

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2,$$

on $\mathbb{R}^4$. Show that the Poisson brackets between (R, S, T) computed in question 9 and the Poisson brackets between (R, S, T) that can be computed using ϖ_j on P_j are the same. This means that we can use the Poisson brackets determined in question 9 in order to define the dynamics on P_j as explained in the next question.

11. Show that on any symplectic manifold P with coordinates $(z_1, \ldots, z_n)$ and symplectic form ω, Poisson brackets act as derivations, i.e.,

$$\{F, G\} = \sum_{j=1}^{n} \{F, z_j\} \frac{\partial G}{\partial z_j} = \sum_{j=1}^{n} \sum_{i=1}^{n} \{z_i, z_j\} \frac{\partial F}{\partial z_i} \frac{\partial G}{\partial z_j}.$$

The above shows that any Poisson bracket (and thus the dynamics) can be determined if we know the Poisson brackets between the coordinates on P.

12. Given a Hamiltonian function F on P_j, i.e., $F = F(j; S, R, T)$, use the Poisson brackets between (R, S, T) to derive the equations of motion. Show that the flow of *any* such F leaves P_j invariant, i.e., if an orbit starts on P_j then it stays there.

13. Consider on $\mathbb{R}^4$ the Hamiltonian function

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{\varepsilon}{4}(p_1^2 - p_2^2 + q_1^2 - q_2^2)^2,$$

where $\varepsilon > 0$. Find the reduced Hamiltonian H_j on $P_j = J^{-1}(j)/\mathbb{S}^1$.

14. Describe completely the dynamics of H_j on P_j. Draw a phase portrait on P_j.