Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 19 May.

12.1 The orthogonal group $O(n, \mathbb{R})$

Let $gl(n, \mathbb{R})$ be the set of all real $n \times n$–matrices. Further define

$$\begin{aligned} Gl(n,\mathbb{R}) &= \{S \in gl(n,\mathbb{R}) \mid \det S \neq 0\} \\ O(n,\mathbb{R}) &= \{S \in gl(n,\mathbb{R}) \mid S^T S = \mathrm{id}\} \\ o(n,\mathbb{R}) &= \{A \in gl(n,\mathbb{R}) \mid A^T = -A\} \\ Sym(n,\mathbb{R}) &= \{A \in gl(n,\mathbb{R}) \mid A^T = A\}. \end{aligned}$$

1. Show that $o(n, \mathbb{R})$ and $Sym(n, \mathbb{R})$ are real vector spaces. Give their dimensions.
2. Show that $Gl(n, \mathbb{R})$ is an n^2–dimensional manifold. Show how $gl(n, \mathbb{R})$ can be regarded as the tangent space $T_{\mathrm{id}} Gl(n, \mathbb{R})$.
3. Let $F : gl(n, \mathbb{R}) \longrightarrow gl(n, \mathbb{R})$ be defined by $F(S) = S^T S$. Show that F is a smooth map and that the image of F is a subset of $Sym(n, \mathbb{R})$.
4. Show that the derivative[1] $D_{\mathrm{id}} F : gl(n, \mathbb{R}) \longrightarrow gl(n, \mathbb{R})$ is given by $D_{\mathrm{id}} F(B) = B^T + B$. What is the rank of this derivative?
5. Show that the rank of the derivative $D_S F : gl(n, \mathbb{R}) \longrightarrow gl(n, \mathbb{R})$ is independent of $S \in O(n, \mathbb{R})$. *Hint:* Use the fact that $O(n, \mathbb{R})$ is a group.
6. Show that $O(n, \mathbb{R})$ is a manifold, also determining its dimension. In what sense can $o(m, \mathbb{R})$ be regarded as the tangent space $T_{\mathrm{id}} O(n, \mathbb{R})$?

12.2 Rigid rotations on the circle, Constant vector fields on the torus

Let $P : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the rigid rotation $\varphi \mapsto \varphi + 2\pi\rho$, where everything is counted mod 2π. For $\rho \in \mathbb{R} \setminus \mathbb{Q}$, show that any orbit $\{\varphi, P(\varphi), P^2(\varphi), \cdots\}$ is dense in $\mathbb{S}^1$.

On $\mathbb{T}^2$ with coordinates (φ_1, φ_2), both counted mod 2π, consider the constant vector field X given by

$$\begin{aligned} \dot{\varphi}_1 &= \omega_1 \\ \dot{\varphi}_2 &= \omega_2. \end{aligned}$$

[1] In another notation, $D_{\mathrm{id}} F = F_{*,\mathrm{id}}$.

Suppose that ω_1 and ω_2 are not rationally related, then show that any integral curve $\{\phi^t \mid t \in \mathbb{R}\}$ of X is dense in $\mathbb{T}^2$. *Hint:* use the Poincaré (return) map of the circle $\varphi_1 = 0$ and the first part of the present exercise.

12.3 Transformations in one degree of freedom

Let $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a given smooth function, with corresponding Hamiltonian vector field X_H. Here we use the standard symplectic structure on $\mathbb{R}^2$. Moreover, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism. Consider both the function $K := H \circ g^{-1}$, together with the associated Hamiltonian vector field X_K, and the transformed vector field $g_\star(X_H)$, defined by $g_\star(X_H)(g(p)) := D_p g X_H(p)$. Show that

$$g_\star(X_H) = \det(Dg) X_K \ .$$

Hint: exploit a coordinate free formulation of the fact that X_H is the Hamiltonian vector field corresponding to H. Discuss the implication for the integral curves of $g_\star(X_H)$ and X_K. Also consider the time-parametrisation of these curves. What happens in the special case that g is canonical?

12.4 A Poincaré–Birkhoff fixed point theorem

Consider the annulus $A := [1, 2] \times \mathbb{S}^1$, with coordinates (I, φ), where φ is counted mod 2π. Consider a smooth, boundary preserving diffeomorphism $T_\varepsilon : A \longrightarrow A$, of the form $T_\varepsilon : (I, \varphi) \mapsto (I, \varphi + 2\pi\rho(I)) + \varepsilon \ (f(I, \varphi, \varepsilon), g(I, \varphi, \varepsilon))$ and such that

- $\rho'(I) \neq 0$, saying that T_ε is a twist-map (for simplicitiy we take ρ increasing);
- $\oint_\gamma I \, \mathrm{d}\varphi = \oint_{T_\varepsilon(\gamma)} I \, \mathrm{d}\varphi$, which means that T_ε is preserving area.

Show that for each rational number p/q, with

$$\rho(1) \leq \frac{p}{q} \leq \rho(2) \ ,$$

in A there exists a periodic point of T_ε, of period q, provided that $|\varepsilon|$ is sufficiently small. *Hint:* Abbreviating $T_\varepsilon^q(I, \varphi) = (I + O(\varepsilon), \Phi_{q,\varepsilon}(I, \varepsilon))$, with $\Phi_{q,\varepsilon}(I, \varphi) = \varphi + 2\pi q \rho(I) + O(\varepsilon)$, consider the equation $\Phi_{q,\varepsilon}(I, \varphi) = \varphi + 2\pi p$, for $p \in \mathbb{Z}$. Use the implicit function theorem in order to obtain a curve $C = \{I = F(\varphi, \varepsilon)\}$ of solutions. Then study the intersection of C and $T_\varepsilon^q(C)$.