Conservative Dynamical Systems

14.1 Normal forms for equilibria of area-preserving maps

Consider an area-preserving map

$$F : \mathbb{R}^2 \to \mathbb{R}^2 : (q,p) \mapsto F(q,p) = (F_1(q,p), F_2(q,p)).$$

Assume that the origin is an elliptic fixed point of F and that in complex coordinates $z = q + ip$, $\bar{z} = q - ip$, F can be written as

$$F(z,\bar{z}) = \Lambda \cdot (z,\bar{z})^T + \sum_{k\geq 2} F^{(k)}(z,\bar{z}),$$

withOC $\Lambda = \mathrm{diag}(\exp(i(\omega+\varepsilon)), \exp(-i(\omega+\varepsilon)))$, $\omega = 2\pi/3$, ε small (but not necessarily positive), and $F^{(k)}$ homogeneous of degree k in z, $\bar{z}$.

Find up to terms of degree 2 the general form of a coordinate transformation

$$(z,\bar{z})^T = \Phi(\zeta,\bar{\zeta}) = (\zeta,\bar{\zeta})^T + \Phi^{(2)}(\zeta,\bar{\zeta}) + \cdots$$

such that in the coordinates ζ, $\bar{\zeta}$, the map F becomes $U(\zeta,\bar{\zeta}) = \Lambda\cdot(\zeta,\bar{\zeta})^T + U^{(2)}(\zeta,\bar{\zeta}) + \cdots$ with

$$\exp(i\omega)U(\zeta,\bar{\zeta}) - U(\exp(i\omega)\zeta, \exp(-i\omega\bar{\zeta})) = 0,$$

up to second degree terms. In particular show that $U^{(2)}(\zeta,\bar{\zeta}) = (u\bar{\zeta}^2, \bar{u}\zeta^2)$ for some $u \in \mathbb{C}$.

Then find a Hamiltonian function $H(\zeta,\bar{\zeta})$ such that $\exp(-i\omega)U$ is the time-1 flow of H up to second degree terms in ζ, $\bar{\zeta}$ and first degree terms in ε. Show that H is an integral of U, i.e., $H \circ U = H$ up to third degree in ζ, $\bar{\zeta}$ and first degree in ε.

Draw for $u = 1$, the level curves of H on $\mathbb{C}$ for ε greater than, equal to, and less than 0. Compare with Figure 239, page 392 in Arnold's book.