Exercise 1 Let $a, b \in \mathbf{R}$, $a < b$. We denote by $L^1(a, b)$ the space of classes of a.e. equal integrable functions, equipped with the norm

$$\|f\|_1 := \int_a^b |f(x)| \, \mathrm{d}x.$$

We recall that $L^1(a, b)$ is a Banach space, i.e., any cauchy sequence in $L^1(a, b)$ has a limit in $L^1(a, b)$. We define the *Sobolev space*

$$W^{1,1}(a, b) := \{u \in \mathcal{D}'(]a, b[) \mid u, u' \in L^1(a, b)\},$$

where u' denotes the distributional derivative of a distribution u. We equip this space with the norm

$$\|u\|_{W^{1,1}(a,b)} := \|u\|_1 + \|u'\|_1.$$

(i) Show that this norm turns $W^{1,1}(a, b)$ into a Banach space.

(ii) Define $f :]a, b[\to \mathbf{C}$ by

$$f(t) := \int_a^t u'(x) \, \mathrm{d}x = \int_a^b \mathbf{1}_{]a,t[}(x) u'(x) \, \mathrm{d}x.$$

Use Lebesgue's Dominated Convergence Theorem to conclude that $f \in C(]a, b[)$.

(iii) Show that for all $u \in W^{1,1}(a, b)$ there is a constant $c \in \mathbf{C}$ so that $u = f + c$. Conclude that any distribution $u \in W^{1,1}(a, b)$ is given by an element of $C(]a, b[)$. (Hint: Show that $f' = u'$ and use Theorem 4.3. The equality $\mathbf{1}_{]a,t[}(x) = \mathbf{1}_{]x,b[}(t)$ may be useful.)

Exercise 2 Suppose $f \in C(\mathbf{R})$ has a distributional derivative $g \in C(\mathbf{R})$. Show that $f \in C^1(\mathbf{R})$ and $f' = g$.