Dynamical Systems 2007

The last two exercises are homework, to be handed in on 19 February.

2.1 Perturbations of an anharmonic oscillator

Determine the bifurcation diagram of the family

$$H_{\lambda,\mu}(x,y) \;=\; \frac{1}{2}y^2 \;+\; \frac{1}{24}x^4 \;+\; \frac{\lambda}{2}x^2 \;+\; \mu x$$

of Hamiltonian systems.

2.2 Planar vector fields

Discuss the following list of planar vector fields. Consider the question whether they are Hamiltonian, gradient or neither of the two. Determine Hamilton or potential functions if possible. What is the connection with *exact* differential equations?

$$\begin{aligned} \dot{x} = y,\ \dot{y} = -x; &\qquad \dot{x} = -y^2,\ \dot{y} = -2xy; \\ \dot{x} = \sin y,\ \dot{y} = \cos x; &\qquad \dot{x} = 2y,\ \dot{y} = 3x; \\ \dot{x} = y - \varepsilon x,\ \dot{y} = -x - \varepsilon y; &\qquad \dot{x} = x,\ \dot{y} = -y. \end{aligned}$$

If you feel like it, you can attack these examples with all the machinery you remember from the course(s) on ODE's: determination of equilibria (stationary solutions) and the corresponding linear parts, a stability analysis, Lyapunov functions, etc.. Also it is useful to draw phase portraits.

2.3 A transcritical bifurcation

Analyse the family

$$H_\lambda(x,y) \;=\; \frac{1}{2}y^2 \;+\; \frac{1}{6}x^3 \;+\; \frac{\lambda}{2}x^2$$

of Hamiltonian systems.

2.4 The Legendre transformation

Let $H(q,p) = \frac{1}{2}p^2 + V(q)$ be the energy function, where $V : \mathbb{R} \longrightarrow \mathbb{R}$ is the potential energy. Compute the Legendre transformation

$$L(q,v) \;:=\; \sup_{p \geq 0} \left(v \cdot p \;-\; H(q,p) \right)$$

and clarify the situation with a figure. The function L obtained this way is the Lagrange function of the system. Show that Hamilton's equations $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$ turn into

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v} \;=\; \frac{\partial L}{\partial q} \;.$$