Dynamical Systems 2007

These two exercises are homework, to be handed in on 19 March.

6.1 Collision time

Two particles P_i with mass m_i, $(i = 1, 2)$ attract each other according to Newton's law with attraction constant f. In the initial position, they are at rest and their distance is $2a$. When do they meet?

6.2 The orthogonal group $O(n, \mathbb{R})$

Let $gl(n, \mathbb{R})$ be the set of all real $n \times n$-matrices. Further define

$$\begin{aligned} Gl(n,\mathbb{R}) &= \{S \in gl(n,\mathbb{R}) \mid \det S \neq 0\}, \\ O(n,\mathbb{R}) &= \{S \in gl(n,\mathbb{R}) \mid S^t S = \mathrm{id}\}, \\ o(n,\mathbb{R}) &= \{A \in gl(n,\mathbb{R}) \mid A^t = -A\}, \\ Sym(n,\mathbb{R}) &= \{A \in gl(n,\mathbb{R}) \mid A^t = A\}. \end{aligned}$$

1. Show that $o(n, \mathbb{R})$ and $Sym(n, \mathbb{R})$ are real vector spaces. Give their dimensions.

2. Show that $Gl(n, \mathbb{R})$ is an n^2-dimensional manifold. Show how $gl(n, \mathbb{R})$ can be regarded as the tangent space $T_{\mathrm{id}} Gl(n, \mathbb{R})$.

3. Let $F : gl(n, \mathbb{R}) \to gl(n, \mathbb{R})$ be defined by $F(S) = S^t S$. Show that F is a smooth map and that the image of F is a subset of $Sym(n, \mathbb{R})$.

4. Show that the derivative $D_{\mathrm{id}} F : gl(n, \mathbb{R}) \to gl(n, \mathbb{R})$[1] is given by $D_{\mathrm{id}} F(B) = B^t + B$. What is the rank of this derivative?

5. Show that the rank of the derivative $D_S F : gl(n, \mathbb{R}) \to gl(n, \mathbb{R})$ is independent of $S \in O(n, \mathbb{R})$. (Hint: Use the fact that $O(n, \mathbb{R})$ is a group.)

6. Show that $O(n, \mathbb{R})$ is a manifold, also determining its dimension. In what sense can $o(m, \mathbb{R})$ be regarded as the tangent space $T_{\mathrm{id}} O(n, \mathbb{R})$?

[1] In another notation, $D_{\mathrm{id}} F = F_{*,\mathrm{id}}$.