Dynamical Systems 2007

The last two exercises are homework, to be handed in on 7 May.

10.1 Geometry of Diophantine numbers

Sketch the set $\Gamma_{\gamma,\tau} := \left\{ \omega \in \mathbb{R}^2 \;\middle|\; |\langle \omega \mid k \rangle| \geq \frac{\gamma}{|k|^\tau} \text{ for all } k \in \mathbb{Z}^2 \backslash \{0\} \right\}$ of (γ, τ)–Diophantine frequencies, where $|k| := |k_1| + |k_2|$.

10.2 Diophantine numbers: A thick Cantor set

In the unit interval $[0, 1]$, for given constants $\gamma > 0$ and $\tau > 2$, consider a subset $D_{\gamma,\tau}$ of Diophantine numbers, defined as follows. We say that $\rho \in D_{\gamma,\tau}$ if for all rational numbers p/q one has

$$\left| \rho - \frac{p}{q} \right| \;\geq\; \frac{\gamma}{q^\tau} \;.$$

Show that $D_{\gamma,\tau}$ is nowhere dense. Also show that the Lebesgue measure of $[0, 1] \setminus D_{\gamma,\tau}$ is of order $O(\gamma)$ as $\gamma \to 0$.

10.3 Lemma of Paley–Wiener

Prove the lemma of Paley–Wiener: a periodic function with Fourier series

$$f(x) \;=\; \sum_{k \in \mathbb{Z}} f_k \, \mathrm{e}^{\mathrm{i}kx}$$

is (real) analytic if and only if the coefficients decay exponentially fast:

$$\bigvee_{M,\eta>0} \;\; \bigwedge_{k \in \mathbb{Z}} \;\; |f_k| \;\leq\; M \cdot \mathrm{e}^{-|k| \cdot \eta} \;.$$

10.4 The spherical pendulum

A spherical pendulum has length ℓ and mass m. Let g be the acceleration of gravity.

1. Give equations of motion with the principle of Hamilton;
2. Determine two (first) integrals, or conserved quantities;
3. Give Hamilton-Jacobi equations for the system, in which the conservation laws are well expressed. Reduce to one degree of freedom (as in the central force field problem).
4. Describe the dynamics of the spherical pendulum in terms of this reduction. First describe the geometry of the invariant level sets defined by the conserved quantities and second characterize the corresponding dynamics. Interprete these findings in the configuration space. Why is this description not complete?