Dynamical Systems 2007

These two exercises are homework, to be handed in on 14 May.

11.1 A Poincaré-Birkhoff fixed point theorem

Consider the annulus $A := [1,2] \times \mathbb{S}^1$, with coordinates (I, φ), where φ is counted mod 2π. Consider a smooth, boundary preserving diffeomorphism $T_\varepsilon : A \to A$, of the form $T_\varepsilon : (I, \varphi) \mapsto (I, \varphi + 2\pi\rho(I)) + \varepsilon(f(I, \varphi, \varepsilon), g(I, \varphi, \varepsilon))$ and such that

- $\rho'(I) \neq 0$, saying that T_ε is a twist-map (for simplicitiy we take ρ increasing);
- $\oint_\gamma I \, d\varphi = \oint_{T_\varepsilon(\gamma)} I \, d\varphi$, which means that T_ε is preserves area.

Show that for each rational number p/q, with

$$\rho(1) \leq \frac{p}{q} \leq \rho(2),$$

in A there exists a periodic point of T_ε, of period q, provided that $|\varepsilon|$ is sufficiently small. (Hint: Abbreviating $T_\varepsilon^q(I, \varphi) = (I + O(\varepsilon), \Phi_{q,\varepsilon}(I, \varepsilon)$, with $\Phi_{q,\varepsilon}(I, \varphi) = \varphi + 2\pi q\rho(I) + O(\varepsilon)$, consider the equation $\Phi_{q,\varepsilon}(I, \varphi) = \varphi + 2\pi p$, for $P \in \mathbb{Z}$. Use the implicit function theorem in order to obtain a curve $C = I = F(\varphi, \varepsilon)$ of solutions. Then study the intersection of C and $T_\varepsilon^q(C)$.)

11.2 A small divisor problem by Sternberg

On $\mathbb{T}^2$, with coordinates (φ_1, φ_2), a vector field X is given, with the following property. If C_1 denotes the circle $C_1 := \{\varphi_1 = 0\}$, then the Poincaré return map $P : C_1 \to C_1$ with respect to X is a rigid rotation $\varphi_2 \mapsto P(\varphi_2) = \varphi_2 + 2\pi\rho$, everything counted mod 2π. From now on we abbreviate $\varphi := \varphi_2$. Let $f(\varphi)$ be the return time of the integral curve connecting the points φ and $P(\varphi)$ in C_1. A priori, f does not have to be constant. The problem now is to construct a(nother) circle C_2, that does have a constant return time. To this purpose let ϕ^t denote the flow of X and express P in terms of ϕ^t and f. Let us look for a circle C_2 of the form

$$C_2 = \{\phi^{\alpha(\varphi)} \mid \varphi \in C_1\}.$$

So the search is for a (periodic) function α and a constant c, such that

$$\phi^c(C_2) = C_2.$$

Rewrite this equation explicitly in terms of α and c. Solve this equation formally in terms of Fourier series. What condition on ρ in general will be needed? Give conditions on ρ, such that for a real analytic function f a real analytic solution α exists.