A dual cusp in a sociological model

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1 Introduction

According to the Cambridge Dictionary, segregation is defined as a policy of separating one group of people from another and treating them differently, especially because of race, sex, or religion. Segregation can be categorized into two types: organized and unorganized. The definition provided refers to organized segregation, which is intentionally enforced, often by institutions or governments. In contrast, unorganized segregation results from individual choices, often influenced by personal preferences.

Because people from a particular population have characteristics that differ from those of other populations, they may exhibit discriminatory behavior toward individuals from other groups. Based on this discriminatory behavior people make decisions about where to live and whom to interact with, resulting in unorganized segregation. This can have both negative and positive effects. On one hand, it can create challenges for integration, particularly for immigrants, as people remain isolated from those in other populations which only deepens the differences. On the other hand, segregation can provide a sense of safety and belonging through stronger connections within one's own group [3].

In this study, we will explore a segregation model introduced by Schelling, known as the Bounded Neighborhood (BN) Model. In this model people from one neighborhood are split into two distinct populations. It assumes that individuals consider the proportion of people from their own population compared to others when deciding whether to remain in a neighborhood or move out. The maximum level of tolerance they have for a particular population is referred to as the tolerance limit. Additionally, it assumes that the original neighborhood is always preferred over others, and the population ratio in outside areas is irrelevant [1]. Analyzing this model illustrates how individual preferences and their resulting actions can lead to segregation. We begin by defining the differential equations of the model, then investigate the equilibrium points and assess their stability. Finally, we use this analysis to derive the bifurcation diagram.

2 Sociological model

We examine the planar model for self-organized segregation, which involves two time-dependent population variables, x and y. Their tolerance schedule is represented by:

$$\dot{x} = P(x, y) := x^2 - x^3 - xy \tag{1}$$

$$\dot{y} = Q(x, y) := \beta y^2 - \alpha\beta y^3 - xy \tag{2}$$

The first equation models the growth of population x as a balance of three factors. The self-reinforcing growth (x^2) indicates that as the number of individuals from x increases, more are likely to join the neighborhood. However, growth is limited by a certain saturation ($-x^3$) and competition with the other population ($-xy$), which slows or halts the expansion of x. Interpretation of the second equation shows a similar pattern. The first term (βy^2) represents the growth of population y, with a higher tolerance of the minority, β, enabling y to grow more easily. The second term ($-\alpha\beta y^3$) limits the growth of y, with α determining the population ratio [2]. A higher α causes quicker saturation, as the upper limit for the y-population is given by $\frac{1}{\alpha}$. By definition, both α and β are positive. The last term ($-xy$) is equal to the last term in the first equation and represents the competition between the y- and the x-population.

2.1 Equilibrium points

We are interested in the steady-state solutions, as they represent the situation where a neighborhood is stable and there is no further movement of individuals from either population. For a solution to be considered a steady-state the right hand side of both differential equations must be equal to zero. In case of the first equation, this implies

$$\begin{aligned}\dot{x} &= 0\\ x^2 - x^3 - xy &= 0\\ x = 0 \quad \text{or} \quad y &= x - x^2.\end{aligned}$$

And for the second equation

$$\begin{aligned}\dot{y} &= 0\\ \beta y^2 - \alpha\beta y^3 - xy &= 0\\ y = 0 \quad \text{or} \quad x &= \beta y - \alpha\beta y^2.\end{aligned}$$

This readily results in three equilibria: $(0,0), (1,0), (0, \frac{1}{\alpha})$, representing cases where neither population is present, only the x-population exists, or only the y-population exists. The fourth equilibrium is more challenging to derive. It corresponds to the intersection of $y = x - x^2$ and $x = \beta y - \alpha\beta y^2$, which occurs when there is a solution to $x = \beta(x - x^2) - \alpha\beta(x - x^2)^2$. Notably, this equation had a trivial solution at $x = 0$ and a second, nontrivial solution. Rewriting the equation leads to the form

$$x = 0 \tag{3}$$

or

$$x^3 - 2x^2 + \frac{1+\alpha}{\alpha}x + \frac{1-\beta}{\alpha\beta} = 0 \tag{4}$$

To solve the nontrivial solution, we observe that since we are dealing with a cubic equation, we must calculate the discriminant D first. When $D < 0$, the equation has three real roots, and when $D > 0$, it has only one real root. The discriminant for an equation of the form $ax^3 + bx^2 + cx + d$ is defined as

$$D = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd,$$

so in this case we have

$$D = (-2)^2 \cdot \left(\frac{1+\alpha}{\alpha}\right)^2 - 4 \cdot \left(\frac{1+\alpha}{\alpha}\right)^3 - 4 \cdot (-2)^3 \cdot \left(\frac{1-\beta}{\alpha\beta}\right) - 27 \cdot \left(\frac{1-\beta}{\alpha\beta}\right)^3 + 18 \cdot -2 \cdot \left(\frac{1+\alpha}{\alpha}\right) \cdot \left(\frac{1-\beta}{\alpha\beta}\right).$$

Solving this results in

$$D = \frac{\beta^2(4-\alpha) + \beta(4\alpha^2 - 18\alpha) + 27\alpha}{\alpha^3\beta^2},$$

which is negative, implying that the cubic equation has three real roots, when

$$\beta_- < \beta < \beta_+$$

with

$$\beta_\pm = \frac{9\alpha - 2\alpha^2 \pm 2\sqrt{\alpha(\alpha-3)^3}}{4-\alpha}$$

The calculation behind this is as follows: $\beta_\pm$ are the values of β for which the discriminant equals zero, i.e. the numerator is zero. Plotting the discriminant against α and β, see Figure 1, we deduce that for $\beta_- < \beta < \beta_+$ the discriminant is negative. We impose the restriction $\alpha > 3$ because, given that $\alpha > 0$, the square root term $\sqrt{\alpha(\alpha-3)^3}$ remains real-valued only for $\alpha > 3$; otherwise, the expression inside the square root becomes negative. Consequently, the square root function is not continuous for $\alpha < 3$. Thus for $\alpha > 3$, $\beta_\pm$ remains real. Additionally, if $\alpha = 3$, β_- and β_+ coincide, meaning there is no β such that $\beta_- < \beta < \beta_+$. Consequently, the discriminant satisfies $D \geq 0$. When $D = 0$, the equation has a double root, implying that instead of three distinct real roots, two roots merge into a double root while the third remains distinct. In the context of dynamical systems, this merging of roots indicates a transition, such as a bifurcation. Lastly, there is a vertical asymptote at $\alpha = 4$.

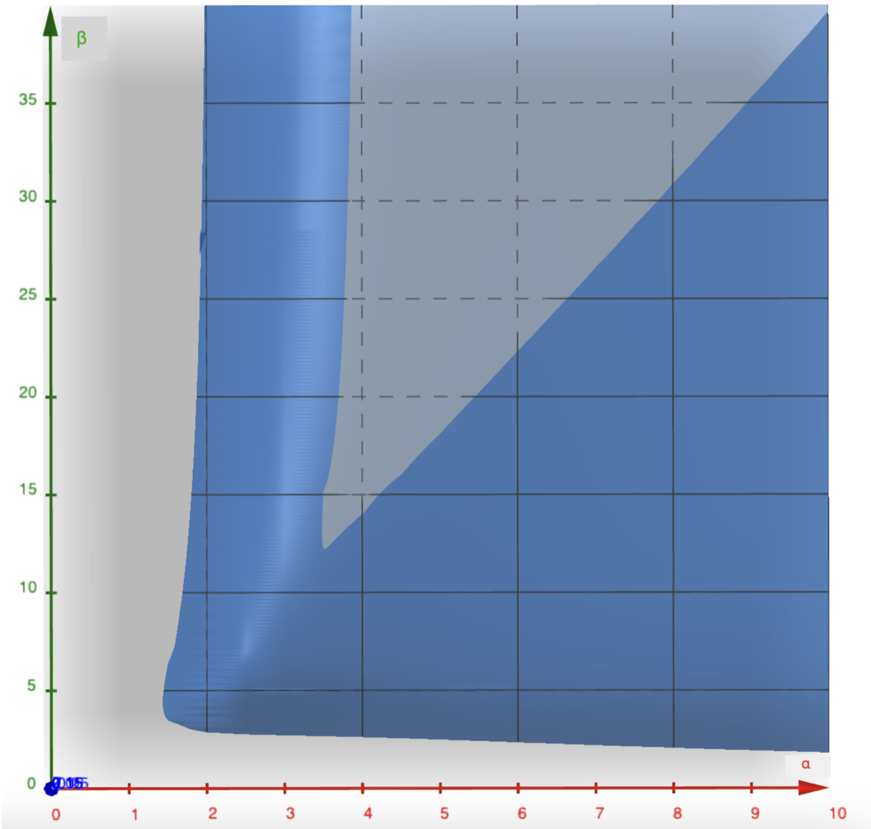


Figure 1: Visualization of the discriminant D along the z-axis, plotted against α and β, with the gray-blue region representing $D < 0$ and the bright blue region representing $D > 0$.

We have now defined two scenarios: when $D > 0$, there are four equilibria, and when $D < 0$, i.e. $\beta_- < \beta < \beta_+$, there are six equilibria. The always-present equilibria are $(0,0), (1,0)$ and $(0, \frac{1}{\alpha})$, while the fourth, and occasionally the fifth and sixth, equilibria can be determined by solving the roots of the cubic equation. We will proceed with analyzing the stability of these equilibria.

2.2 Stability

To investigate stability we construct the Jacobian matrix of the planar system and substitute the steady states into it. The Jacobian matrix is given by

$$J = \begin{pmatrix} 2x - 3x^2 - y & -x \\ -y & 2\beta y - 3\alpha\beta y^2 - x \end{pmatrix}.$$

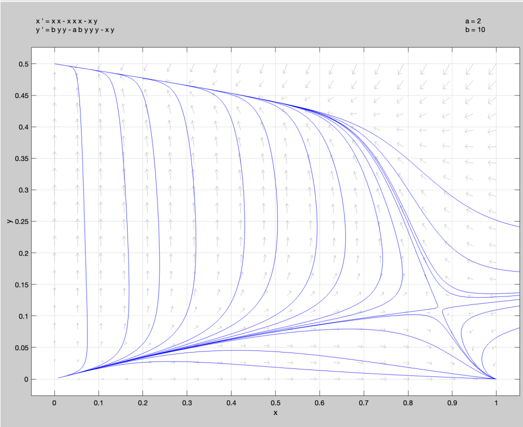
Substituting $(0,0)$ gives the eigenvalues $\lambda_{1,2} = 0$, which implies that this is a nonhyperbolic equilibrium and that the stability of this equilibrium depends on the type of perturbation. Additional second order analysis by H. Hanßmann and A. Momin (2024) [3] reveals that $(0,0)$ is stable for $\beta < 1$. However, it is important to note that this equilibrium represents the scenario of an empty neighborhood, making it less significant for practical consideration. Evaluating the second equilibrium $(1,0)$, which represents the scenario where the entire neighborhood is occupied by the x-population, gives the eigenvalues $\lambda_{1,2} = -1$, indicating that this is a stable node. Similarly, the equilibrium $(0, \frac{1}{\alpha})$, where the entire neighborhood is occupied by the y-population, is also a stable node, as $J(0, \frac{1}{\alpha})$ yields the eigenvalues $\lambda_1 = -\frac{1}{\alpha}$ and $\lambda_2 = -\frac{\beta}{\alpha}$.

Next, we examine the remaining possible equilibria, which correspond to the solutions of the cubic equation. Determining the stability of these equilibria is significantly more challenging compared to the others. However, numerical analysis reveals that the stability of the equilibria is not influenced by their exact location. In the article "A Dynamical Systems Model of Unorganized Segregation" [2], D.J. Haw and S.J. Hogan derived the equilibria for each (α, β) and evaluated the corresponding Jacobian. By plotting the determinant of the Jacobian as a function of (α, β) they discovered that in the region with three equilibria, one is stable while the other two are saddle points. Outside this region, where only a single real solution exists, the equilibrium is a saddle point. In both cases, the regions of attraction for the stable equilibrium points are bounded by the stable manifolds of either two or one saddle points, respectively. This observation by D.J. Haw and S.J. Hogan is validated by the phase portraits we generated. Specifically, for $D < 0$, we observe three mixed equilibrium points: two saddles and one stable node (see Figure 2(c)). Conversely, for $D > 0$, there is one mixed equilibrium point, a saddle (see Figures 2(a) and 2(g)). Evaluating the stability of the equilibrium points brings us to the final step in analyzing the planar model: deriving a bifurcation diagram.

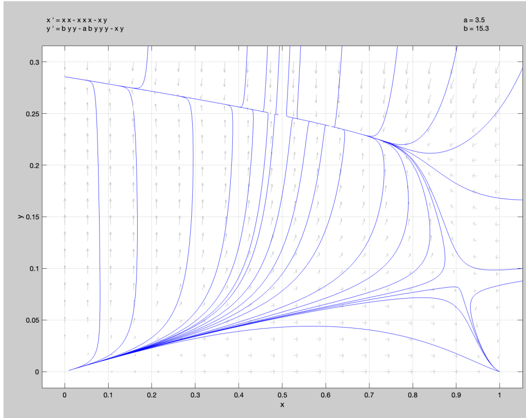
2.3 Bifurcation diagram

Bifurcations happen when the system loses stability, meaning that near a bifurcation point, even a small perturbation in parameters can change the stability of equilibrium points. We have established that inside the region where $\beta_- < \beta < \beta_+$, there are six equilibrium points, while outside this region there are four. This indicates that bifurcations occur along the lines β_- and β_+. To illustrate this, we generated six phase portraits with varying values of (α, β).

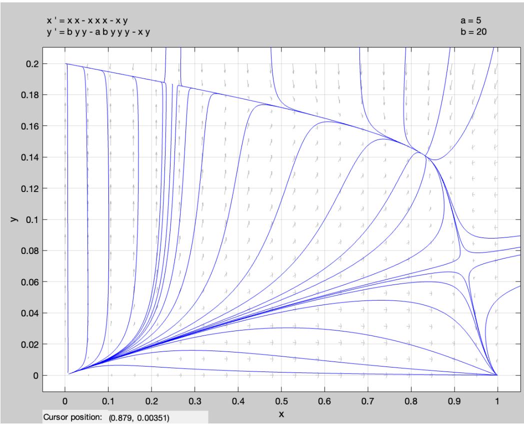
Figure 2: Phase portraits corresponding to different points (α, β).



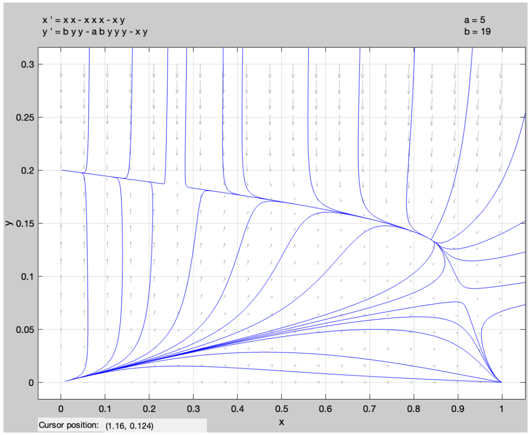
(a) Outside the region where $D < 0$ (2,10).



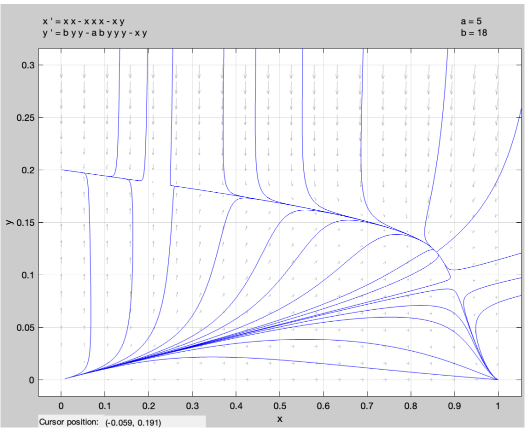
(b) On the bifurcation curve β_+ (3.5,15.3).



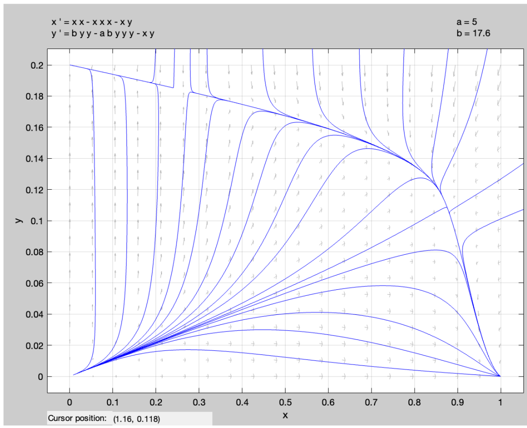
(c) Inside the region where $D < 0$ (5,20).



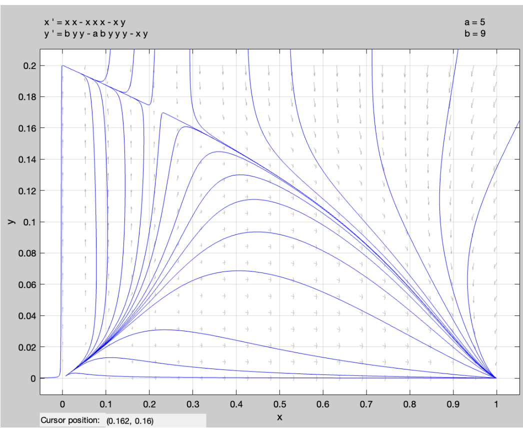
(d) Inside the region where $D < 0$ (5,19).



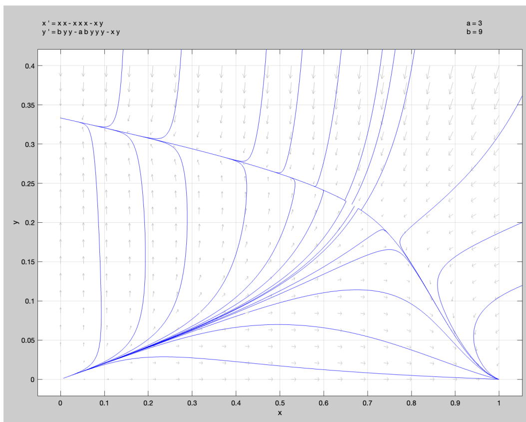
(e) Inside the region where $D < 0$ (5,18).



(f) On the bifurcation curve β_- (5,17.6).



(g) Outside the region where $D < 0$ (5,9).



(h) At the apex (3,9).

The phase portraits illustrate the arising of a node and a saddle during the first fold bifurcation, occurring as the bifurcation curve β_+ is crossed from the region where $D > 0$. Within the region where $D < 0$, we observe six equilibrium points as expected: three stable nodes, two saddles, and an unstable node at $(0, 0)$. When approaching β_-, the node and the rightmost saddle move toward each other, which can be seen in Figure 2(d) and 2(e). Along the bifurcation curve, they converge and eventually vanish, as shown in the second last diagram outside the $D < 0$ region. This behavior corresponds to the second fold bifurcation. At the apex, where $(\alpha, \beta) = (3, 9)$ and β_- intersects with β_+, a transition occurs, leading to a change in the existence and stability of the equilibria. Here, the two saddle points and the node merge, leaving only a single saddle point. This phenomenon is called a dual-cusp bifurcation. In terms of our system this means that a change in tolerance limits (β) and population ratio (α) causes reorganization of the population distribution. As individuals adjust their decisions based on their tolerance limits it effects the stability and existence of equilibria. Specifically, the merging of a stable node with two saddle points into a single saddle implies that parameter changes can drive large-scale segregation. With only two stable nodes remaining—one dominated by the x-population and the other by the y-population—the system evolves toward complete segregation.

Identifying and analyzing the bifurcations present in the sociological model enables us to construct the corresponding bifurcation diagram. We present the bifurcation diagram developed by H. Hanßmann and A. Momin (2024) [3], with the x-population represented on the vertical axis. In this diagram, the green surface represents the saddle points of the system, while the red surface corresponds to the stable nodes. The surfaces meet along the blue lines, indicating the occurrence of the fold bifurcations. At the apex, where the two blue lines meet, the dual-cusp bifurcation takes place.

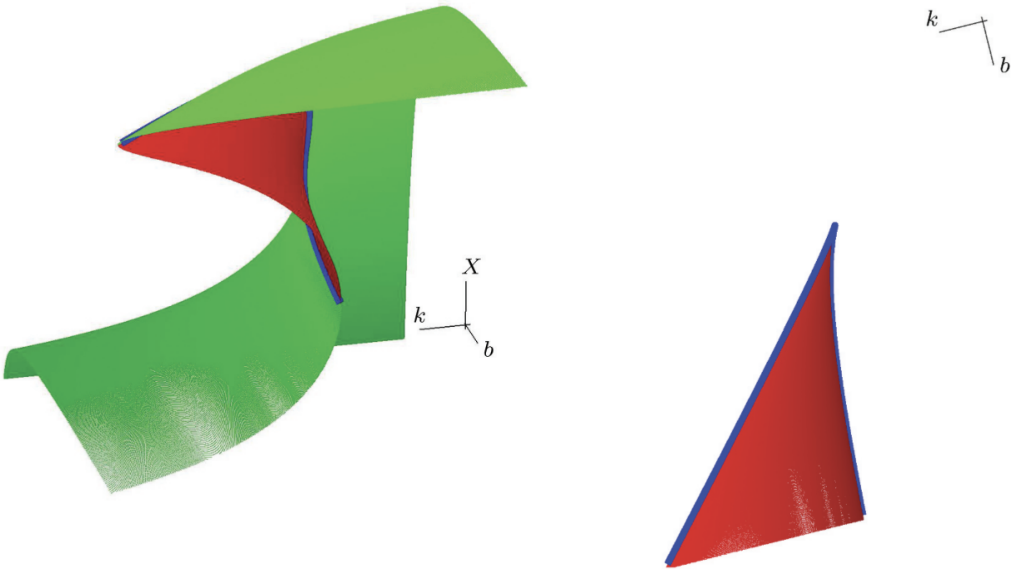
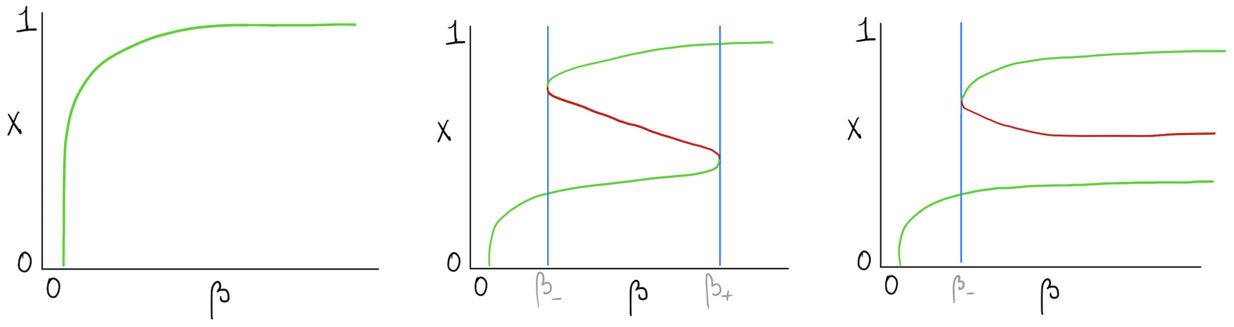


Figure 3: Bifurcation diagram, where k corresponds to α, b to β and X to the x-population [3].

To clarify the bifurcation diagram, we consider three separate cases for the value of α. As previously noted, since α is positive, the restriction $\beta_- < \beta < \beta_+$ holds only when $a > 3$, and we also observed the presence of a vertical asymptote at $\alpha = 4$. Therefore, we consider the cases $0 < \alpha < 3$, $3 < \alpha < 4$, and $\alpha > 4$, which correspond to moving from right to left along the k-axis in the left-hand image of Figure 3. For each case, we will identify the projection of the 3D bifurcation diagram onto 2D intersections, as is shown in Figure 4, while maintaining the same color scheme. Green represents the saddles, red denotes the stable node, and the blue lines indicate the bifurcation curves. The equilibrium points $(0, 0)$, $(1, 0)$, and $\left(0, \frac{1}{\alpha}\right)$ are excluded from this analysis. We focus exclusively on the equilibrium points obtained by solving the cubic equation.

Figure 4: Intersections of the 3D bifurcation surface with β, X-planes for constant α.



(a) $0 < \alpha < 3$.

(b) $3 < \alpha < 4$.

(c) $\alpha > 4$.

Figure 4(a) shows no bifurcation points, which aligns with the restriction that β_- and β_+ are only defined for $\alpha > 3$. Since we are in the region where $D > 0$, corresponding to the blue region in Figure 1, we expect only one solution: the saddle node.
In Figure 4(b), we observe two bifurcation curves. In this interval for α, the vertical asymptote is not yet present. As shown in Figure 1, for α in this interval, we cross both bifurcation curves as β increases. Between these curves, we are in the region where $D < 0$, resulting in three equilibrium points: two saddles and one stable node. The fold bifurcations occur along the blue lines, specifically at β_-, where two equilibrium points arise, and at β_+, where two equilibrium points merge and vanish. Finally, Figure 4(c) illustrates that for $\alpha > 4$, we only cross β_- because β_+ has a vertical asymptote at $\alpha = 4$, as also seen in Figure 1. Here, we observe one fold bifurcation; the arising of two equilibrium points. After crossing β_-, we enter the region where $D < 0$, which persists as β increases. In this region, there are three equilibrium points: two saddles and one stable node.

3 Conclusion

In this paper, we explored Schelling's Bounded Neighborhood Model, beginning with an introduction to the topic of segregation and the model itself. We then interpreted the differential equations describing the system and calculated the equilibrium points. Our findings revealed that the system can have four, five or six equilibrium points. There are always two stable nodes at $(1, 0)$ and $(0, \frac{1}{\alpha})$, and one unstable node at $(0, 0)$ for $\beta > 1$. Additional equilibrium points, determined by solving the cubic equation, may consist of either one saddle point or two saddles and one stable node. The distinction between these cases is determined by the discriminant of the cubic equation. We identified the curves where the number of equilibria changes, noticing two fold bifurcations. The first fold bifurcation results in the emergence of a saddle point and a stable node, while the second fold bifurcation leads to the merging and vanishing of a saddle and stable node. Finally, we described a dual cusp bifurcation at the apex, where two saddles and one stable node merge, leaving only a saddle node. We summarized all of these findings in a bifurcation diagram.

References

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[2] D.J. Haw and S.J. Hogan, A dynamical systems model o f unorganized segregation. J. Math. Sociol. 42(3):113–127 (2018) [Pages 113–121]

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