2. Introduction to DFIELD5

A first order ordinary differential equation has the form

$$x' = f(t, x).$$

To solve this equation we must find a function $x(t)$ such that

$$x'(t) = f(t, x(t)), \qquad \text{for all } t.$$

This means that at every point $(t, x(t))$ on the graph of x, the graph must have slope equal to $f(t, x(t))$.

We can turn this interpretation around to give a geometric view of what a differential equation is, and what it means to solve the equation. At each point (t, x), the number $f(t, x)$ represents the slope of a solution curve through this point. Imagine, if you can, a small line segment attached to each point (t, x) with slope $f(t, x)$. This collection of lines is called a *direction line field*, and it provides the geometric interpretation of a differential equation. To find a solution we must find a curve in the plane which is tangent at each point to the direction line at that point.

Admittedly, it is difficult to visualize such a direction field. This is where the MATLAB routine `dfield5` demonstrates its value[1]. Given a differential equation, it will plot the direction lines at a large number of points — enough so that the entire direction line field can be visualized by mentally interpolating between the field elements. This enables the user to get some geometric insight into the solutions of the equation.

Starting DFIELD5

To see `dfield5` in action, enter `dfield5` at the MATLAB prompt. After a short wait, a new window will appear with the label DFIELD5 Setup. Figure 2.1 shows how this window looks on a PC running Windows 95. The appearance will differ slightly depending on your computer, but the functionality will be the same on all machines.

The DFIELD5 Setup window is an example of a MATLAB *figure window*. A figure window can assume a variety of forms as will soon become apparent. In a MATLAB session there will always be one command window open on your screen and perhaps a number of figure windows as well.

You will notice that the equation $x' = x^2 - t$ is entered in the edit box entitled "The differential equation" of the DFIELD5 Setup window. There is also an edit box for the independent variable and several edit boxes are available for parameters. Note the default values in the "display window," in which the independent variable t is set to satisfy $-2 \le t \le 10$, and the dependent variable x is set to satisfy $-4 \le x \le 4$. At the bottom of the DFIELD5 Setup window there are three buttons labeled Quit, Revert, and Proceed.

We will describe this window in detail later, but for now click the button with the label Proceed. After a few seconds another window will appear, this one labeled DFIELD5 Display. An example of this window is shown in Figure 2.2.

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[1] The MATLAB function `dfield5` is not distributed with MATLAB. To discover if it is installed properly on your computer enter `help dfield5` at the MATLAB prompt. If it is not installed, see the Preface for instructions on how to obtain it.

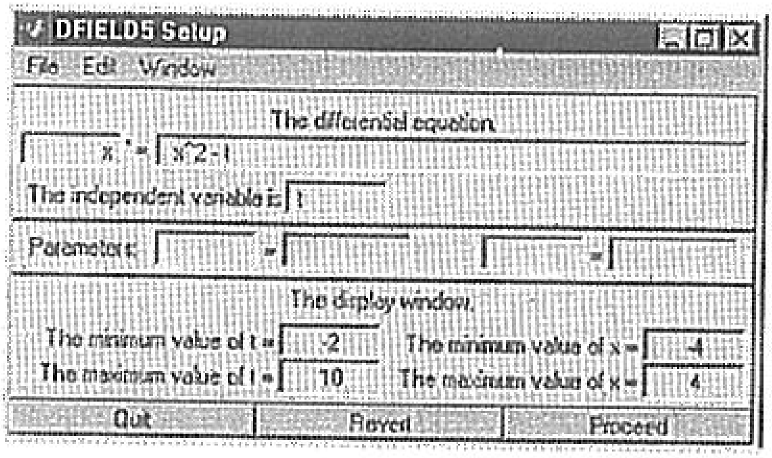

Figure 2.1. The Setup window for `dfield5`.

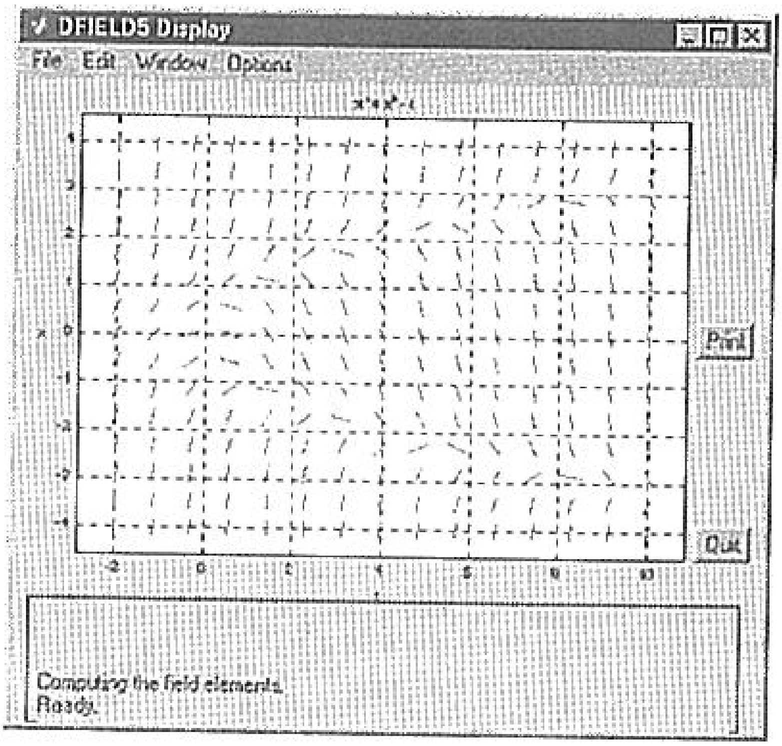

Figure 2.2. The display window for `dfield5`.

The most prominent feature of the DFIELD5 Display window is a rectangular grid labeled with the differential equation $x' = x^2 - t$ on the top, the independent variable t on the bottom, and the dependent variable x on the left. The dimensions of this rectangle are slightly larger than the rectangle specified in the DFIELD5 Setup window so as to accomodate the extra space needed by the direction field lines.

Inside this rectangle is a grid of points, 20 in each direction, for a total of 400 points. At each such point with coordinates (t, x) there is shown a small line segment centered at (t, x) with slope equal to $x^2 - t$.

There is a pair of buttons on the DFIELD5 Display window: **Quit** and **Print**. There are several menus: File, Edit, Window, and Options. Below the direction field there is a message window through which `dfield5` will communicate with the user. Note that the last line of this window contains the word "Ready," indicating that `dfield5` is ready to follow orders.

A *solution curve* of a differential equation $x' = f(t, x)$ is the graph of a function $x(t)$ which solves the differential equation. Computing and plotting a solution curve is very easy using `dfield5`. Choose an initial point for the solution, move the mouse to that point, and click the mouse button. The computer will compute and plot the solution through the selected point, first in the direction in which the independent variable is increasing (the "Forward" direction), and then in the opposite direction (the "Backward" direction).

After computing and plotting several solutions, the display should look something like that shown Figure 2.3.

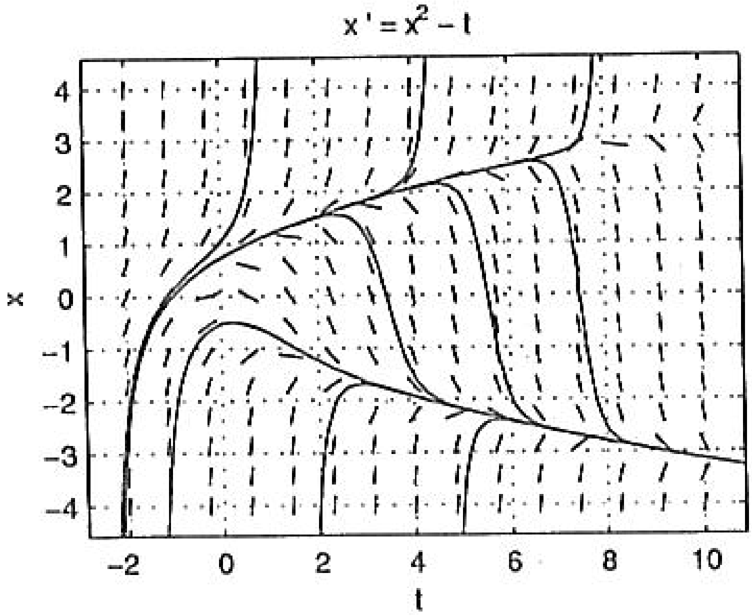

Figure 2.3. Several solutions of the ODE $x' = x^2 - t$.

`Dfield5` also allows you to plot several solutions at once. Select **Options→Plot several solutions**[2] and note that the mouse cursor changes to "crosshairs" when positioned over the direction field. Select several initial conditions for your solutions by clicking the mouse button at several different locations in the direction field. When you are finished selecting initial conditions, position the mouse crosshairs

[2] The notation **Options→Plot several solutions** signifies that you should select "Plot several solutions" from the Options menu. We will use this notation frequently in this manual.

over the direction field and press the **Enter** or **Return** key on your keyboard. Solution trajectories will emanate from the initial conditions you selected with the mouse. Wait for the word "Ready" to appear in the message window before continuing with further computation.

You can also choose the direction of solution trajectories: Forward, Backward, or Both. Select **Options→Solution direction→Forward**, then click your mouse in the direction field and note the effect this option has on solution directions. Experiment further with **Options→Solution direction→Back** and **Options→Solution direction→Both**.

Initial Value Problems

The differential equation $x' = x^2 - t$ has infinitely many solutions. This is suggested by the fact that you get a solution no matter where you click in the display window. Sometimes you need to find a particular solution of a differential equation, a solution that satisfies some initial condition. The differential equation with initial condition

$$x' = f(t, x), \quad x(t_0) = x_0,$$

is called an *initial value problem*. In this case, the dependent variable in the equation $x' = f(t, x)$ is x and the independent variable is t. However, you are free to choose other letters to represent the dependent and/or independent variables[3].

Example 1. *Use* `dfield5` *to find the solution of the initial value problem*

$$y' + y = 3 + \cos x, \quad y(0) = 1,$$

on the interval $0 \le x \le 20$.

The dependent variable in this example is y and the independent variable is x. Solve the differential equation for y'.

$$y' = -y + 3 + \cos x$$

The equation $y' = -y + 3 + \cos x$ is now in the form $y' = f(x, y)$, where $f(x, y) = -y + 3 + \cos x$.

Return to the DFIELD5 Setup window and select Edit→Clear all. Options on the Edit menu clear particular regions of the DFIELD5 Setup window and each of these options possesses a keyboard accelerator. Enter the left and right sides of the differential equation $y' = -y+3+\cos x$, the independent variable, and define the display window in the DFIELD5 Setup window as shown in Figure 2.4.

Should your data entry become hopelessly mangled, click the **Revert** button to restore the original entries. The initial value problem $y' = -y + 3 + \cos x$, $y(0) = 1$, contains no parameters, so leave the parameter fields in the DFIELD5 Setup window blank. Click the **Proceed** button to transfer the information in the DFIELD5 Setup window to the DFIELD5 Display window and start the computation of the direction field.

Choosing the initial point for the solution curve with the mouse is convenient, but sometimes it is necessary to start a solution at an exact initial condition. This is difficult to accomplish with the mouse. Instead, in the DFIELD5 Display window, select **Options→Solution direction→Both**, then **Options→Keyboard input**. Enter the initial condition, $y(0) = 1$, as shown in Figure 2.5.

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[3] MATLAB is case-sensitive. Thus, the variable `Y` is completely different from the variable `y`.

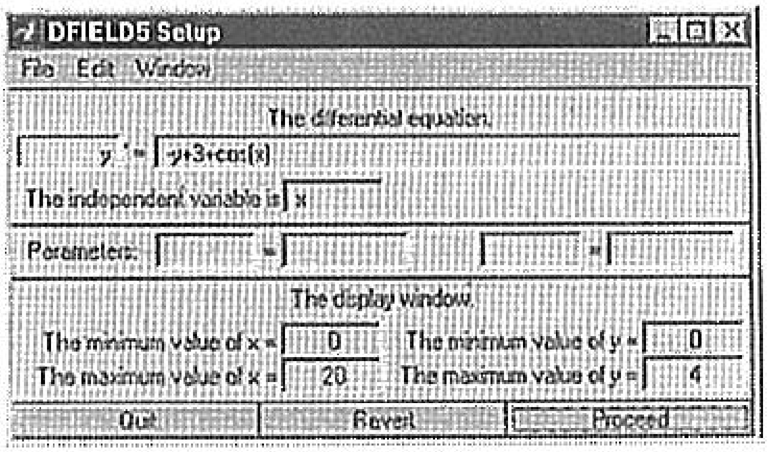

Figure 2.4. Setup window for $y' = -y + 3 + \cos x$.

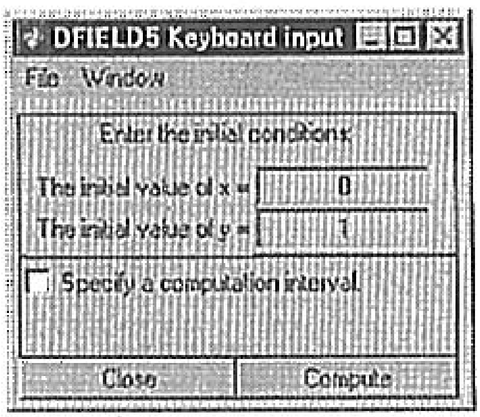

Figure 2.5. The initial condition $y(0) = 1$ starts the solution trajectory at $(0, 1)$.

Click the Compute button in the DFIELD5 Keyboard input window to compute the trajectory shown in Figure 2.6.

Note that you can specify a computation interval by clicking the "Specifying a computation interval" checkbox in the DFIELD5 Keyboard Input window (See Figure 2.5). Simply click the checkbox then fill in the starting and ending times of the solution interval desired. For example, start a solution trajectory with initial condition $y(0) = 0$, but set the computation interval so that $0 \le x \le 2\pi$. Try it!

Existence and Uniqueness

It would be comforting to know in advance that a solution of an initial value problem exists, especially if you are about to invest a lot of time and energy in an attempt to find a solution. A second (but no less important) question is uniqueness: is there only one solution? Or does the initial value problem have more than one solution? Fortunately, existence and uniqueness of solutions has been thoroughly

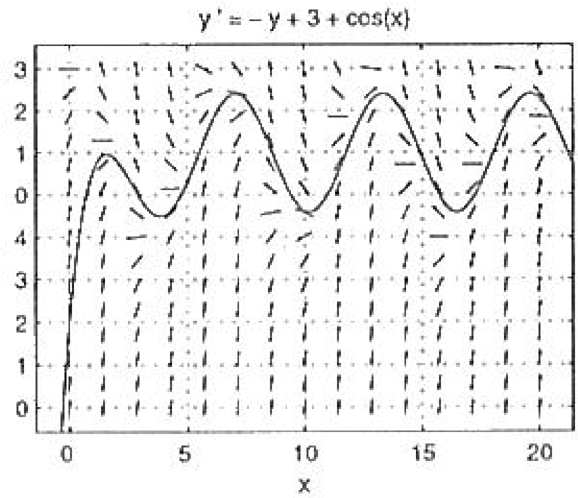

Figure 2.6. Solution of $y' + y = 3 + \cos x$, $y(0) = 1$.

examined and there is a beautiful theorem that we can use to answer these questions.

Theorem 1. *Suppose that the function $f(t, x)$ is defined in the rectangle R defined by $a \le t \le b$ and $c \le x \le d$. Suppose also that f and $\partial f/\partial x$ are both continuous in R. Then, given any point $(t_0, x_0) \in R$, there is one and only one function $x(t)$ defined for t in an interval containing t_0 such that $x(t_0) = x_0$, and $x' = f(t, x)$. Furthermore, the function $x(t)$ is defined both for $t > t_0$ and for $t < t_0$, at least until the graph of x leaves the rectangle R through one of its four edges* [4].

Example 2. *Use* `dfield5` *to sketch the solution of the initial value problem*

$$\frac{dx}{dt} = x^2, \quad x(0) = 1.$$

Set the display window so that $-2 \le t \le 3$ and $-4 \le x \le 4$.

Enter the differential equation $dx/dt = x^2$, the independent variable t, and the display window ranges $-2 \le t \le 3$ and $-4 \le x \le 4$ in the DFIELD5 Setup window. Click **Proceed** to compute the direction field. Select **Options→ Keyboard input** in the DFIELD5 Display window and enter the initial condition $x(0) = 1$ in the DFIELD5 Keyboard input window. If all goes well, you should produce an image similar to that in Figure 2.7.

The differential equation $dx/dt = x^2$ is in the form $x' = f(t, x)$, with $f(t, x) = x^2$. Note also that $f(t, x) = x^2$ and $\partial f/\partial x = 2x$ are continuous on the rectangle R defined by $-2 \le t \le 3$ and $-4 \le x \le 4$. Therefore, Theorem 1 states that the solution shown in Figure 2.7 is unique. Use the mouse

[4] The notation $\partial f/\partial x$ represents the *partial derivative of f with respect to x*. Suppose, for example, that $f(t, x) = x^2 - t$. To find $\partial f/\partial x$, think of t as a constant and differentiate with respect to x to obtain $\partial f/\partial x = 2x$. Similarly, to find $\partial f/\partial t$, think of x as a constant and differentiate with respect to t to obtain $\partial f/\partial t = -1$.

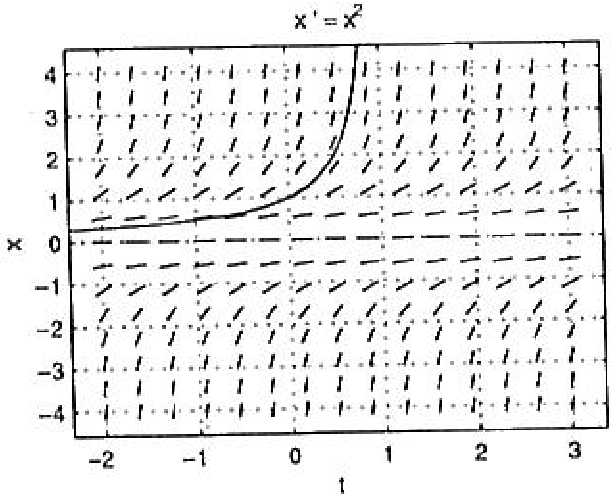

Figure 2.7. The solution of $dx/dt = x^2, x(0) = 1$ is unique.

to experiment. You should not be able to find any other solution trajectories that pass through the point (0, 1).

Theorem 1 makes no guarantees about extending the solution in time (forward or backward). For example, does the solution in Figure 2.7 move upward and to the right forever? Or, does it reach positive infinity in a finite amount of time? This question cannot be determined by `dfield5` alone. However, the differential equation $dx/dt = x^2$ is separable, yielding a family of solutions $x(t) = 1/(C-t)$, where C is an arbitrary constant. If you substitute the initial condition $x(0) = 1$ into the equation $x(t) = 1/(C-t)$, then $1 = 1/(C-0)$, producing the constant $C = 1$ and the solution $x(t) = 1/(1-t)$. The solution equation $x(t) = 1/(1-t)$ implies that $\lim_{t \to 1^-} x(t) = +\infty$. Mathematicians like to say that the solution "blows up[5]." In this particular case, if the independent variable t represents time (in seconds), then the solution trajectory reaches positive infinity before one second of time elapses.

Example 3. *Consider the differential equation*

$$\frac{dx}{dt} = x^2 - t.$$

Sketch solutions with initial conditions $x(2) = 0$, $x(3) = 0$, and $x(4) = 0$. Determine whether or not these solution curves intersect in the display window defined by $-2 \le t \le 10$ and $-4 \le x \le 4$.

Enter the differential equation $x' = x^2 - t$, the independent variable t, and the display window ranges $-2 \le t \le 10$ and $-4 \le x \le 4$ in the DFIELD5 Setup window. Click **Proceed** to transfer this information and begin computation of the direction field in the DFIELD5 Display window. Select **Options→Keyboard input** in the DFIELD5 Display window and compute solutions for each of the initial conditions $x(2) = 0$, $x(3) = 0$, and $x(4) = 0$. If all goes well, you should produce an image similar to that in Figure 2.8.

[5] The graph of the solution reaches infinity (or negative infinity) in a finite time period.

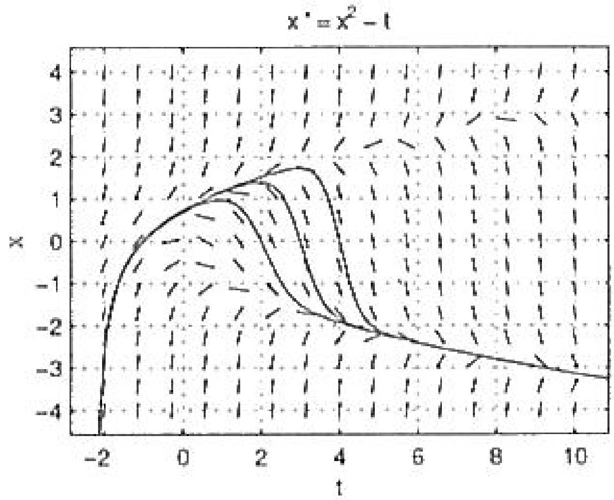

Figure 2.8. Do the trajectories intersect?

When you compare the ODE $x' = x^2 - t$ with general form $x' = f(t, x)$, you see that $f(t, x) = x^2 - t$ is continuous on the display window defined by $-2 \le t \le 10$ and $-4 \le x \le 4$. Moreover, $\partial f/\partial x = 2x$ is also continuous on this display window. In Figure 2.8, it appears that the solution trajectories merge into one trajectory near the point $(8, -2.7)$ (or perhaps even sooner). However, Theorem 1 guarantees that solutions cannot cross or meet in the display window of Figure 2.8.

Let's do some analysis with the zoom tools in `dfield5`. On a PC[6], you would select **Edit→Zoom in** in the **DFIELD5 Display** window, then use the left mouse button and single-click in the **DFIELD5 Display** window near the point $(8, -2.7)$. Additional "zooms" require that you revisit **Edit→Zoom in** before left-clicking the mouse button to zoom. There is a faster way to zoom in that is platform dependent. For example, on a PC or UNIX box, click the right mouse button at the zoom point (or control-click the left mouse button at the zoom point). On a Macintosh, option-click the mouse button at the zoom point. After performing numerous zooms (results may vary on your machine), each time clicking the right mouse button on the trajectory near $t = 8$, some separation in the trajectories begins to occur, as shown in Figure 2.9. Without Theorem 1, we might have mistakenly assumed that the trajectories merged into one trajectory.

Qualitative Analysis

Suppose that you model a population with a differential equation. If you want to use your model to predict the exact population in three years, then you will need to do one of two things: (1) find the solution using analysis, or (2) use a numerical routine to find the solution. However, if your only interest is what happens to the population after a long period of time, it may not be necessary to find an analytical or numerical solution. A qualitative approach might be more appropriate.

[6] Mouse actions are platform dependent in `dfield5`. See the front and back covers of this manual for a summary of mouse actions on various platforms.

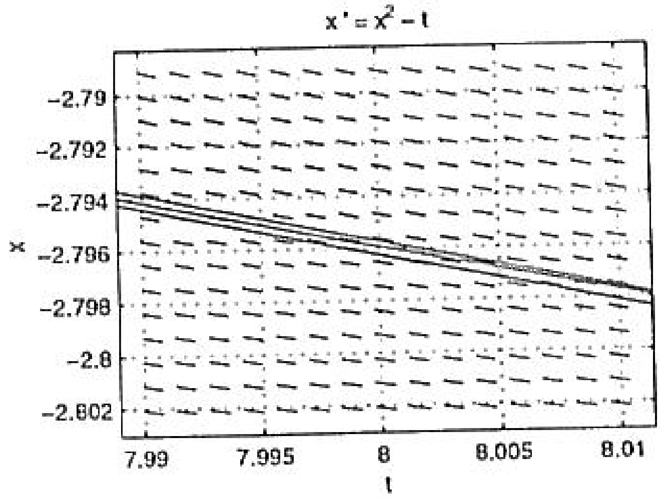

Figure 2.9. The trajectories don't merge or cross.

Example 4. *Consider the logistic population model*

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P. \tag{2.1}$$

Let $P(t)$ represent the population at time t. Let $k = 0.75$ and $N = 10$ and suppose that the initial population at time zero is given by $P(0) = 2$, where the population is measured in millions of people. What will happen to this population over a long period of time?

If you plot the right hand side of equation (2.1) versus P (i.e., plot $k(1 - P/N)P$ versus P), the result is the inverted parabola seen in Figure 2.10. Set $k(1 - P/N)P$ equal to zero to find that the graph crosses the P-axis in Figure 2.10 at $P = 0$ and $P = N$.

It is easily verified that $P(t) = N$ is a solution of $dP/dt = k(1 - P/N)P$ by substituting $P(t) = N$ into each side of the differential equation and simplifying. Similarly, the solution $P(t) = 0$ is easily seen to satisfy the differential equation.

Although the solutions $P(t) = 0$ and $P(t) = N$ might be considered "trivial" since they are constant functions, they are by no means trivial in their importance. In fact, the solutions $P(t) = 0$ and $P(t) = N$ are called *equilibrium solutions*. For example, if $P(t) = N$, then the growth rate dP/dt of the population is zero and the population remains at $P(t) = N$ forever. Similarly, if $P(t) = 0$, the growth rate dP/dt equals zero and the population remains at $P(t) = 0$ for all time.

When the graph of $k(1 - P/N)P$ (which equals to dP/dt) falls below the P-axis in Figure 2.10, then $dP/dt < 0$ and the first derivative test implies that $P(t)$ is a decreasing function of t. On the other hand, when the graph of $k(1 - P/N)P$ rises above the P-axis in Figure 2.10, then $dP/dt > 0$ and $P(t)$ is an increasing function of t. These facts are summarized on the *phase line* below the graph in Figure 2.10. The information on the phase line indicates that a population beginning between 0 and N million people has to increase to the equilibrium value of N million people.

The dependent variable of the differential equation $dP/dt = k(1 - P/N)P$ is P and the independent variable is t; k and N are called *parameters*. Return to the DFIELD5 Setup window and enter the

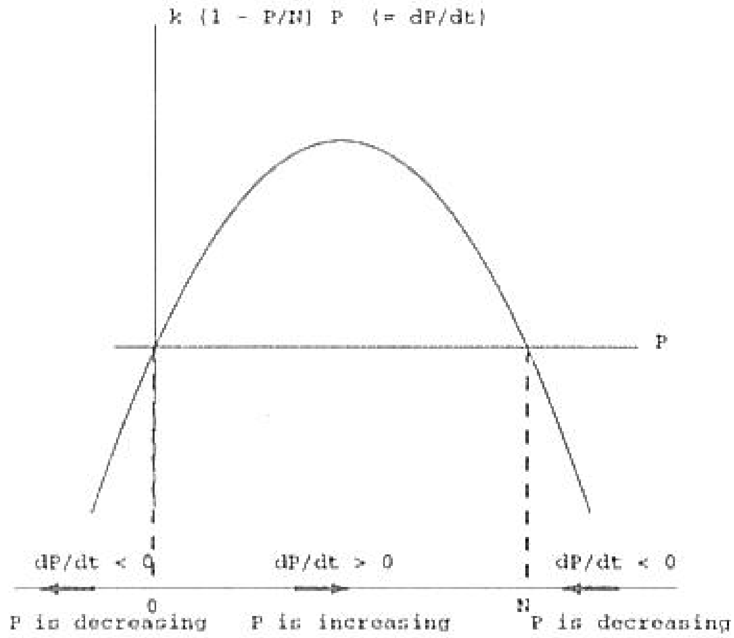

Figure 2.10. The plot of $k(1 - P/N)P$ versus P.

differential equation, the independent variable t, and the parameters $k = 0.75$ and $N = 10$ as shown in Figure 2.11. Set the display window ranges to $0 \le t \le 10$ and $-4 \le P \le 15$ (See Figure 2.11). Click the **Proceed** button to transfer the information in the DFIELD5 Setup window to the DFIELD5 Display window and start the computation of the direction field.

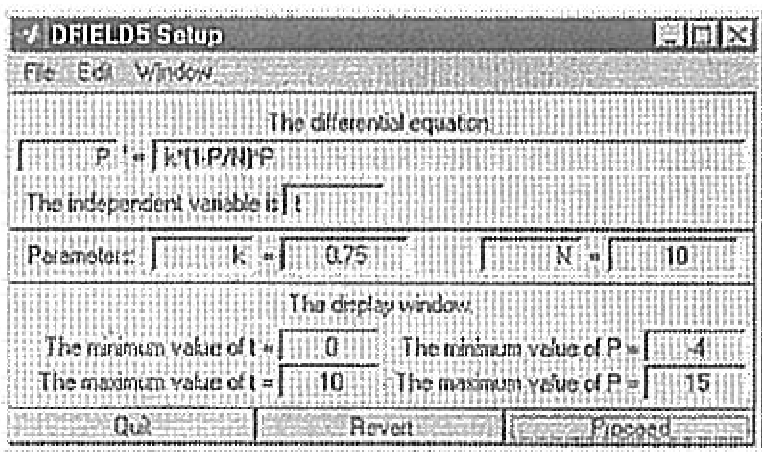

Figure 2.11. Setup window for $dP/dt = k(1 - P/N)P$.

Select **Options→Keyboard input** in the DFIELD5 Display window and start solution trajectories with initial conditions $(0, 0)$ and $(0, 10)$. In Figure 2.12, note that these equilibrium solutions are horizontal lines. Select **Options→Solution direction→Forward** and **Options→Show the phase line** in the DFIELD5 Display window. `Dfield5` aligns the phase line from Figure 2.10 in a vertical direction at the left edge of the direction field in the DFIELD5 Display window. Select **Options→Keyboard input** to begin the solution with initial condition $P(0) = 2$ and note the action of the animated point on the phase line. As the solution trajectory in the direction field approaches the horizontal line $P = 10$, the point on the phase line approaches equilibrium point $P = 10$ on the phase line, as shown in Figure 2.12. It would appear that a population with initial conditions and parameters described in the original problem statement will have to approach 10 million people with the passage of time.

Experiment with some other initial conditions. Note that solutions beginning a little above or a little below the equilibrium solution $P = 10$ tend to move back toward this equilibrium solution with the passage of time. This is why the solution $P = 10$ is called a *stable* equilibrium solution. However, solutions beginning a little above or a little below the equilibrium solution $P = 0$ tend to move away from this equilibrium solution with the passage of time. The solution $P = 0$ is called an *unstable* equilibrium solution. You can review the results of our experiments in Figure 2.12.

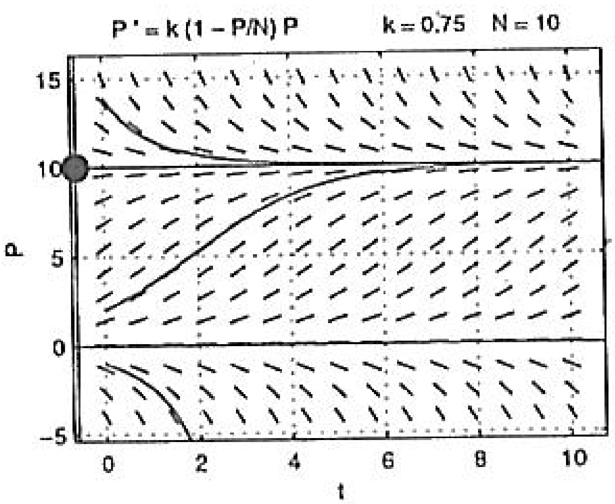

Figure 2.12. Note the vertical phase line at the left of the window.

Zooming and Stopping

It is rare that a solution takes an inordinate amount of time to draw, but should this occur you can halt computation by clicking the **Stop** button in the DFIELD5 Display window. You may have to click the **Stop** button twice: once for the forward direction, a second time to stop the computation in the backward direction. For example, enter the equation `y'=exp(2*t)*cos(y)` in the DFIELD5 Setup window, set the independent variable as `t`, then set the display window so that $-1 \leq t \leq 6$ and $0 \leq y \leq 3$. Select **Options→Solution direction→Both** and use the Keyboard input window to start a solution with initial condition $y(0) = 0$. Note that the forward solution begins to stall. Click the **Stop** button to halt the forward solution and note that the backward solution completes rather quickly.

You can use the mouse and/or menu selections to "zoom in" or "zoom back" in the DFIELD5 Display window. PC users can drag a "zoom box" around the solution of `y'=exp(2*t)*cos(y)` by depressing the right mouse button, then dragging the mouse. Once the zoom box is drawn around the area of interest, release the mouse button and the contents of the zoom box will be magnified to full size of the display window.

`Dfield5` allows you to revisit any of your zoom windows. Select **Edit→Zoom back** in the DFIELD5 Display window. This will open the DFIELD5 Zoom back dialog box pictured in Figure 2.13. Select the zoom window you wish to revisit and click the **Zoom** button.

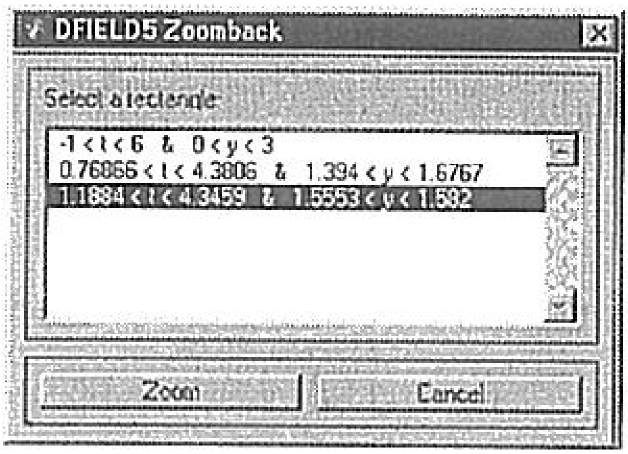

Figure 2.13. Select a zoom window and click the Zoom button.

Using MATLAB While DFIELD5 is Open

You can use MATLAB commands to plot to the DFIELD5 Display window; or, you can open another figure window by typing `figure` at the MATLAB prompt. Future plotting commands will be directed to this window, as long as you do not click the mouse while the cursor is in another window. Remember, all graphics commands will usually be sent to the *active* figure window, which is always the most recently visited window.

Example 5. *The following data represent the temperature T of a potato after t minutes in an oven. Use Newton's Law of Cooling and* `dfield5` *to fit a curve to the data.*

t (min)	0	1	2	3	4	5	6	7	8	9	10
T (°F)	60	213	306	362	397	417	430	438	442	445	447

Newton determined that the rate at which an object warms (or cools) is proportional to the difference between the temperature of the object and its surrounding medium. Consequently,

$$\frac{dT}{dt} = k(A - T), \quad T(0) = 60,$$

where T is the temperature of the potato in degrees Fahrenheit, t is the time in minutes, A is the temperature of the oven, and k is a proportionality constant. Enter the information shown in Figure 2.14 in the DFIELD5 Setup window.

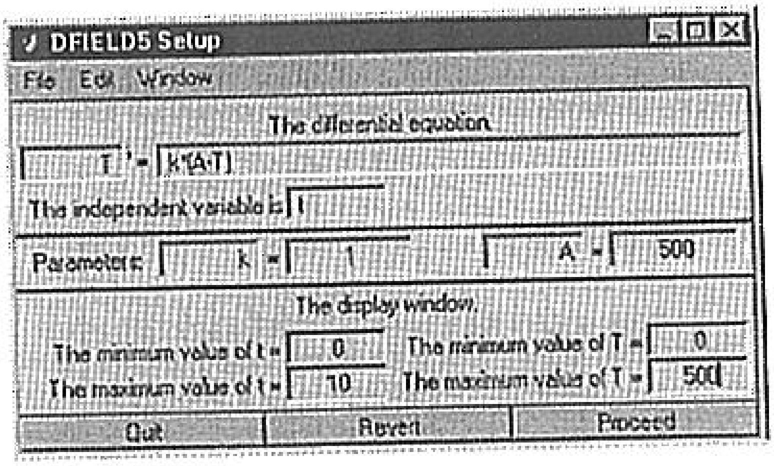

Figure 2.14. Setting up $dT/dt = k(A - T)$.

Note that we have made some wild guesses for the parameters k and A. Click on the Proceed button to process this information and compute the direction field in the DFIELD5 Display window. This also makes the DFIELD5 Display window the current figure window.

Enter the following commands in the MATLAB command window at the MATLAB prompt:

```
>> t=[0,1,2,3,4,5,6,7,8,9,10];
>> T=[60,213,306,362,397,417,430,438,442,445,447];
>> plot(t,T,'o')
```

Finally, select **Options→Keyboard input** and plot the trajectory with initial condition $T(0) = 60$. Experiment with the parameters k and A and the DFIELD5 Keyboard input dialog box until you find a solution that passes through each of the data points. *Hint: The parameters* $k = 0.5$ *and* $A = 450$ *produced the image in Figure 2.15.*

Changing the Size and Appearance of the Display Window

Some people prefer to use a *vector field* rather than a direction field in the DFIELD5 Display window. In a vector field, a vector is attached to each point instead of the line segment used in a direction field. The vector has its base at the point in question, its direction is the slope, and the length of the vector reflects the magnitude of the derivative.

To change the direction field in the DFIELD5 Display window to a vector field, select **Options→ Windows settings** from the Options menu in the DFIELD5 Display window. This will open the DFIELD5 Windows settings dialog box (see Figure 2.16).

Note the three radio buttons that allow you to choose between a line field, a vector field, or no field at all. Select one of these and then click the Change settings button to note the affect on the direction field. Should you select vectors as your option, note that the length of each arrow reflects the *relative magnitude* of the derivative at that point.

There is an edit box in the DFIELD5 Window settings dialog that allows the user to choose the

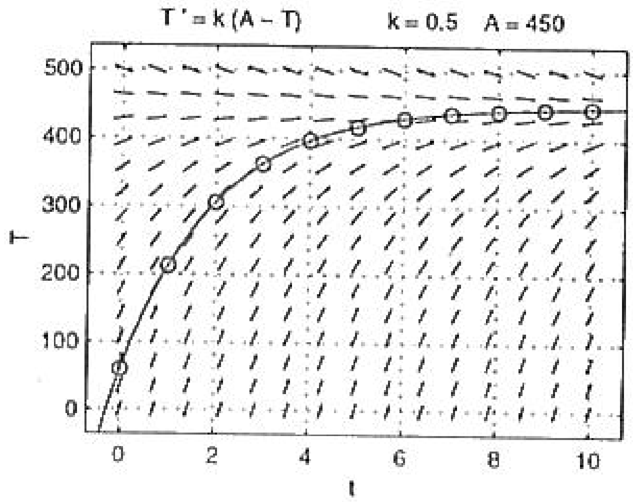

Figure 2.15. Plotting in the DFIELD5 Display window.

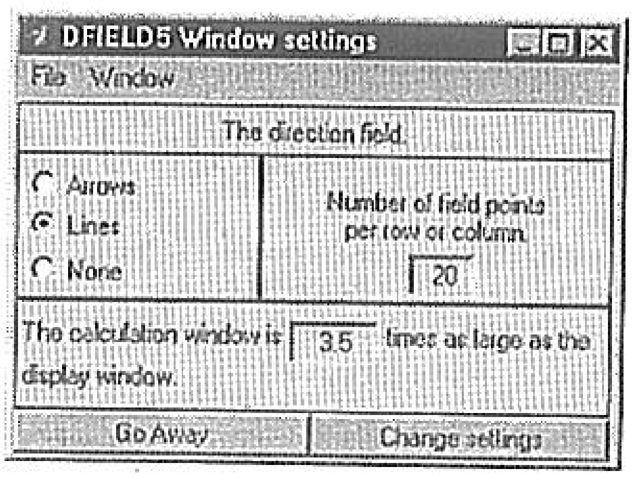

Figure 2.16. DFIELD5 Window settings.

number of field points displayed. The default is 20 points in each row and in each column. Change this entry to 10, hit Enter, then click the Change settings button to note the affect on the direction field.

The design of `dfield5` includes the definition of two windows: the DFIELD5 Display window and the *calculation window*. When you start `dfield5`, the calculation window is 3.5 times as large as the display window in each dimension. The computation of a solution will stop only when the solution curve leaves the calculation window. This allows some room for zooming to a larger display window without having incomplete solution curves. It also allows for some reentrant solution curves, i.e. those which leave the display window and later return to it.

The third item in the DFIELD5 Window settings dialog box controls the relative size of the calculation window. It can be given any value greater than or equal to 1. The smaller this number, the faster dfield5 will compute solutions, and the more likely that reentrant solutions will be lost. If you have a slower computer, and if you are not going to be zooming to a larger display window, it is not completely unreasonable to set this value to 1, but you have to realize that with this choice all reentrant solutions will be lost. A better choice would be 2 or 2.5. The default value of 3.5 seems to meet most needs.

Personalizing the Display Window

Sometimes when you are preparing a display window for printing, you plot a solution curve you wish were not there. There are two methods which allow you to erase items in the DFIELD5 Display window. **Edit→Erase all solutions** is self explanatory. **Edit→Delete a graphics object** is much more flexible. It will allow you to delete individual solution curves. Simply select **Edit→Delete a graphics object**, then click the mouse on the solution curve you wish to delete. You can use **Edit→Delete a graphics object** to delete any graphics object in the DFIELD5 Display window, including text.

There are three text elements which are part of the display window. These are the `title` at the top, the `xlabel` at the bottom, and the `ylabel` at the left. These are given default values by `dfield5` using the information entered into the DFIELD5 Setup window, but they can be changed at any time. Should you want the title and axes labels to reflect the content of your project, try something like this.

```
>> xlabel('Time (years)')
>> ylabel('Frank''s cell count')
>> title('Petri Dish History')
```

Notice that each of the three commands takes as a parameter a text string which is contained between two single quotes. You can use any text string you think is appropriate.

There is a special problem with the `ylabel` string in this example. You might think that it should read `'Frank's cell count'`. The problem is that the prime `'` is being used both to denote an apostrophe and to indicate the beginning or end of the text string. MATLAB needs a different way to designate a prime internal to a text string, and it uses the double prime `''` to do just this.

It is also possible to add text at arbitrary points in the DFIELD5 Display window. Select **Edit→Enter text on the Display Window**, enter the desired text in the Text entry dialog box, then click the OK button. Use the mouse to click at the lower left point of the position in the figure window where you want the text to appear. It can easily happen that your placement of the text does not please you. If so, remove the text using **Edit→Delete a graphics object**, then try again.

Printing, Quitting, and Using Clipboards

The easiest way to print the display window is to click the **Print** button in the DFIELD5 Display window.

If you have previous experience using MATLAB, you realize that the display window, or more generally, the current figure window, can be printed by entering `print` at the MATLAB prompt in the command window. You can also use the print command at the command line to save the contents of a figure window to a graphics file. For example,

```
>> print -deps junk.eps
```

will save the DFIELD5 Display window as an encapsulated postscript file in the current directory. The command

```
>> print -deps -noui junk.eps
```

will also save DFIELD5 Display window, but without the graphical user interface objects such as buttons and message windows. You might want to click on File on the DFIELD5 Display menu and select Page Position. This will open a dialog box that will allow the resizing of the figure window (manually or with a mouse). After you have resized the page position, retry the command `print -deps -noui junk.eps`. For a full set of options on MATLAB's `print` command, type `help print` at the MATLAB prompt.

In the Macintosh and PC-Windows version of MATLAB, the contents of the display window can be copied into a clipboard, and from there into other documents in the standard ways. For example, if you are using a PC with the Windows 95 operating system, Alt+PrintScrn (Holding down the Alt key while pressing PrintScrn) will copy the current figure window to the clipboard. Consult your computer's operating manual for details on copying and pasting images to and from your clipboard.

Always wait until the word "Ready" appears in the `dfield5` message window before you try to do anything else with `dfield5` or MATLAB. When you want to quit `dfield5`, the best way is to use the Quit buttons found on the DFIELD5 Setup or on the DFIELD5 Display windows. Either of these will close all of the `dfield5` windows in an orderly manner, and it will delete the temporary files that `dfield5` creates in order to do its business.

Exercises

For the differential equations in problems 1–3, perform each of the following tasks.

i) Print out the direction field for the differential equation with the display window defined by $t \in [-5, 5]$ and $y \in [-5, 5]$. You might consider increasing the number of field points to 25 in the DFIELD5 Window settings dialog box. On this printout, sketch with a pencil as best you can the solution curves through the initial points $(t_0, y_0) = (0, 0)$, $(-2, 0)$, $(-3, 0)$, $(0, 1)$, and $(4, 0)$. Remember that the solution curves must be tangent to the direction lines at each point.

ii) Use `dfield5` to plot the same solution curves to check your accuracy. Turn in both versions.

1. $y' = y^2 - t^2$
2. $y' = 2ty/(1 + y^2)$
3. $y' = y(2 + y)(2 - y)$
4. Use `dfield5` to plot a few solution curves to the equation $x' + x\sin(t) = \cos(t)$. Use the display window defined by $x \in (-10, 10)$ and $t \in (-10, 10)$.
5. Use `dfield5` to plot the solution curves for the equation $x' = 1 - t^2 + \sin(tx)$ with initial values $x = -3, -2, -1, 0, 1, 2, 3$ at $t = 0$. Find a good display window by experimentation.
6. Consider the differential equation

$$y' + 4y = 8.$$

a) Use `dfield5` to plot a few solutions with different initial points on the display window bounded by $-5 \le t \le 5$ and $-1 \le y \le 5$. In particular, plot the solution curve with initial condition $y(1) = 2$ (use **Options→Keyboard input**). Print out the Figure Window and turn it in as part of this assignment.

b) What do you conjecture is the limiting behavior of the solutions of this differential equation as $t \to \infty$?

c) Find the general analytic solution to this equation.

d) Verify the conjecture you made in part b), or if you no longer believe it, make a new conjecture and verify that.

7. Consider the differential equation

$$(1+t^2)y' + 4ty = t.$$

a) Use **dfield5** to calculate and plot a few solutions with different initial points. (Use the display window defined by $t \in [-5, 5]$ and $y \in [-5, 5]$.) In particular, plot the solution curve with initial condition $y(1) = 1/4$ (use **Options→Keyboard input**). Print out the Figure Window and turn it in as part of this assignment.

b) What do you conjecture is the limiting behavior of the solutions of this differential equation as $t \to \infty$?

c) Find the general analytic solution to this equation.

d) Verify the conjecture you made in part b), or if you no longer believe it, make a new conjecture and verify that.

For Problems 8–11 we will consider a certain lake which has a volume of $V = 100$ km^3. It is fed by a river at a rate of r_i km^3/year, and there is another river which is fed by the lake at a rate which keeps the volume of the lake constant. In addition, there is a factory on the lake which introduces a pollutant into the lake at the rate of p km^3/year. This means that the rate of flow from the lake into the outlet river is $(p + r_i)$ km^3/year. Let $x(t)$ denote the volume of the pollutant in the lake at time t, and let $c(t) = x(t)/V$ denote the concentration of the pollutant.

8. Show that, under the assumption of immediate and perfect mixing of the pollutant into the lake water, the concentration satisfies the differential equation $c' + ((p + r_i)/V)c = p/V$.

9. Suppose that $r_i = 50$, and $p = 2$.

a) Suppose that the factory starts operating at time $t = 0$, so that the initial concentration is 0. Use **dfield5** to plot the solution.

b) It has been determined that a concentration of over 2% is hazardous for the fish in the lake. Approximately how long will it take until this concentration is reached? You can "zoom in" on the **dfield5** plot to enable a more accurate estimate.

c) What is the limiting concentration? About how long does it take for the concentration to reach a concentration of 3.5%?

10. Suppose the factory stops operating at time $t = 0$, and that the concentration was 3.5% at that time. Approximately how long will it take before the concentration falls below 2%, and the lake is no longer hazardous for fish? Notice that $p = 0$ for this exercise.

11. Rivers do not flow at the same rate the year around. They tend to be full in the Spring when the snow melts, and to flow more slowly in the Fall. To take this into account, suppose the flow of our river is

$$r_i = 50 + 20\cos(2\pi(t - 1/3)).$$

Our river flows at its maximum rate one-third into the year, i.e., around the first of April, and at its minimum around the first of October.

a) Setting $p = 2$, and using this flow rate, use **dfield5** to plot the concentration for several choices of initial concentration between 0% and 4%. (You might have to reduce the relative error tolerance in **Options->Solver settings**, perhaps to 5×10^{-12}, or `5e-12`.) How would you describe the behavior of the concentration for large values of time?

b) It might be expected that after settling into a steady state, the concentration would be greatest when the flow was smallest, i.e., around the first of October. At what time of year does it actually occur?

12. Use **dfield5** to plot several solutions to the equation $z' = (z - t)^{5/3}$. (**Hint:** Notice that when $z < t$, $z' < 0$, so the direction field should point down, and the solution curves should be decreasing. You might have difficulty getting the direction field and the solutions to look like that. If so read the section in Chapter 1 on complex arithmetic, especially the last couple of paragraphs.)

A differential equation of the form $dx/dt = f(x)$, whose right-hand side does not explicitly depend on the independent variable t, is called an *autonomous* differential equation. For example, the logistic model in Example 4 was autonomous. For the autonomous differential equations in Problems 13–17, perform each of the following tasks. Note that the first three tasks are to be performed without the aid of technology.

i) Set the right-hand side of the differential equation equal to zero and solve for the equilibrium points.

ii) Plot the graph of the right-hand side of each autonomous differential equation versus x, as in Figure 2.10. Draw the phase line below the graph and indicate where x is increasing or decreasing, as was done in Figure 2.10.

iii) Use the information in parts (i) and (ii) to draw sample solutions in the xt plane. Be sure to include the equilibrium solutions.

iv) Check your results with **dfield5**. Again, be sure to include the equilibrium solutions.

v) If x_0 is an equilibrium point, i.e., if $f(x_0) = 0$, then $x(t) = x_0$ is an equilibrium solution. It can be shown that if $f'(x_0) < 0$, then every solution curve that has an initial value near x_0 converges to x_0 as $t \to \infty$. In this case x_0 is called a *stable* equilibrium point. If $f'(x_0) > 0$, then every solution curve that has an initial value near x_0 diverges away from x_0 as $t \to \infty$, and x_0 is called an *unstable* equilibrium point. If $f'(x_0) = 0$, no conclusion can be drawn about the behavior of solution curves. In this case the equilibrium point may fail to be either stable or unstable. Apply this test to each of the equilibrium points.

13. $x' = \cos(\pi x)$, $x \in [-3, 3]$.
14. $x' = x(x-2)$, $-\infty < x < \infty$.
15. $x' = x(x-2)^2$, $-\infty < x < \infty$.
16. $x' = x(x-2)^3$, $-\infty < x < \infty$.
17. $x' = x(1 + e^{-x} - x^2)$, $-1 \le x \le 2$. In this case you will not be able to solve explicitly for all of the equilibrium points. Instead, turn the problem around. Use **dfield5** to plot some solutions, and from that information calculate approximately where the equilibrium points are, and determine the type of each. Check your estimate with this code:

```
f=inline('x*(1+exp(-x)-x^2)')
z=fzero(f,1)
f(z)
```

The logistic equation is

$$P' = \alpha P\left(1 - \frac{P}{N}\right).$$

The quantities α and N are the parameters of the equation. Usually the parameters are constants, and in that case the kind of analysis carried out in Problems 13–17 shows that for any solution $P(t)$ which has a positive initial value we have $P(t) \to N$ as $t \to \infty$. For this reason N is called the *carrying capacity* of the system.

There are some models in which the carrying capacity is not a constant, but depends on time. In cases like this can we say anything about the relationship between the long term behavior of the solutions and the carrying capacity? For the carrying capacities in Problems 18–21 you are to examine the long term behavior of solutions, especially in comparison to the carrying capacity. In particular:

a) Use **dfield5** to plot several solutions to the equation. (It is up to you to find a display window that is appropriate to the problem at hand.)

b) Based on the plot done in part a), describe the asymptotic behavior of the solutions to the equation. In particular, compare this asymptotic behavior to the asymptotic behavior of N. It might be helpful to plot N on Display Window produced in part a). In the first two cases the solutions will be asymptotic to a constant. In the other two the solutions will be asymptotic to a function. You are not expected to find that function explicitly, but you should be able to describe it qualitatively.

c) It is possible to find the solutions to these equations explicitly (except, perhaps, for the evaluation of an integral). Find these solutions. (**Hint:** Look up Bernoulli's equation in your textbook.)

Consider the equation in the following four cases:

18. $N(t) = 1$, $\alpha = 1$. This case is the standard logistic equation. Consequently, whatever the initial population, we expect that $P(t) \to N$ as $t \to \infty$. This case is here for comparison with the other three.

19. $N(t) = 1 - \frac{1}{2}e^{-t}$, $\alpha = 1$. In this case $N(t)$ is monotone increasing, and $N(t)$ is asymptotic to 1. This might model a situation of a human population where, due to technological improvement, the availability of resources is increasing with time, although ultimately limited.

20. $N(t) = 1 + t$, $\alpha = 1$. Again $N(t)$ is monotone increasing, but this time it is unbounded, although of a very simple nature. This might model a situation of a human population where, due to technological improvement, the availability of resources is steadily increasing with time, and therefore the effects of competition are becoming less severe. Note: The latest version of **dfield5** supports the entry of expressions in the parameter windows. After entering the parameter N in the DFIELD5 Setup window, set N equal to **1+t**.

21. $N(t) = 1 - \frac{1}{2}\cos(2\pi t)$, $\alpha = 1$. This is perhaps the most interesting case. Here the carrying capacity is periodic in time with period 1, which might be considered to be one year. This might model a population of insects or small animals that are affected by the seasons. You will notice that the asymptotic behavior as $t \to \infty$ reflects the behavior of N. The solution does not tend to a constant, but nevertheless all solutions have the same asymptotic behavior for large values of t. In particular, you should take notice of the location of the maxima of N and of P. You can use the "zoom in" option to get a better picture of this. *Note: It is interesting to superimpose the plot of the carrying capacity $N(t) = 1 - \frac{1}{2}\cos(2\pi t)$ on the solution in the DFIELD5 Display window. Try the following at the* MATLAB *prompt:* `t=linspace(a,b)`*, where* `a` *and* `b` *are the bounds for* `t` *in the DFIELD5 Display window. Then follow with* `N=1-1/2*cos(2*pi*t)` *and* `plot(t,N)`.

22. Despite the seeming generality of the uniqueness theorem, there are initial value problems which have more than one solution. Consider the differential equation $y' = \sqrt{|y|}$. Notice that $y(t) \equiv 0$ is a solution with the initial condition $y(0) = 0$. (Of course by $\sqrt{|y|}$ we mean the nonnegative square root.)

   a) This equation is separable. Use this to find a solution to the equation with the initial value $y(t_0) = 0$ assuming that $y \ge 0$. You should get the answer $y(t) = (t - t_0)^2/4$. Notice, however, that this is a solution only for $t \ge t_0$. Why?

   b) Show that the function
$$y(t) = \begin{cases} 0, & \text{if } t < t_0; \\ (t - t_0)^2/4, & \text{if } t \ge t_0; \end{cases}$$
   is continuous, has a continuous first derivative, and satisfies the differential equation $y' = \sqrt{|y|}$.

   c) For any $t_0 \ge 0$ the function defined in part b) satisfies the initial condition $y(0) = 0$. Why doesn't this violate the uniqueness part of the theorem?

   d) Find another solution to the initial value problem in a) by assuming that $y \le 0$.

   e) You might be curious (as were the authors) about what **dfield5** will do with this equation. Find out. Use the rectangle defined by $-1 \le t \le 1$ and $-1 \le y \le 1$ and plot the solution of $y' = \sqrt{|y|}$ with initial value $y(0) = 0$. Also, plot the solution for $y(0) = 10^{-50}$ (the MATLAB notation for 10^{-50} is `1e-50`). Plot a few other solutions as well. Do you see evidence of the non-uniqueness observed in part c)?

An important aspect of differential equations is the dependence of solutions on initial conditions. There are two points to be made. First, we have a theorem which says that the solutions are continuous with respect to the initial conditions. More precisely,

Theorem. *Suppose that the function $f(t, x)$ is defined in the rectangle R defined by $a \le t \le b$ and $c \le x \le d$. Suppose also that f and $\frac{\partial f}{\partial x}$ are both continuous in R, and that*
$$\left|\frac{\partial f}{\partial x}\right| \le L \quad \text{for all } (t, x) \in R.$$
If (t_0, x_0) and (t_0, y_0) are both in R, and if
$$\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0 \end{aligned} \quad \text{and} \quad \begin{aligned} y' &= f(t, y) \\ y(t_0) &= y_0 \end{aligned}$$

then for $t > t_0$

$$|x(t) - y(t)| \leq e^{L(t-t_0)}|x_0 - y_0|$$

as long as both solution curves remain in R.

Roughly, the theorem says that if we have initial values that are sufficiently close to each other, the solutions will remain close, at least if we restrict our view to the rectangle R. Since it is easy to make measurement mistakes, and thereby get initial values off by a little, this is reassuring.

For the second point, we notice that although the dependence on the initial condition is continuous, the term $e^{L(t-t_0)}$ allows the solutions to get exponentially far apart as the interval between t and t_0 increases. That is, the solutions can still be extremely sensitive to the initial conditions, especially over long t intervals.

23. Consider the differential equation $x' = x(1 - x^2)$.
    a) Verify that $x(t) \equiv 0$ is the solution with initial value $x(0) = 0$.
    b) Use `dfield5` to find approximately how close the initial value y_0 must be to 0 so that the solution $y(t)$ of our equation with that initial value satisfies $y(t) \leq 0.1$ for $0 \leq t \leq t_f$, with $t_f = 2$. You can use the display window $0 \leq t \leq 2$, and $0 \leq x \leq 0.1$, and experiment with initial values in the **Options→ Keyboard input** window, until you get close enough. Do not try to be too precise. Two significant figures is sufficient.
    c) As the length of the t interval is increased, how close must y_0 be to 0 in order to insure the same accuracy? To find out, repeat part b) with $t_f = 4, 6, 8$, and 10.

It is clear from the results of the last problem that the solutions can be extremely sensitive to changes in the initial conditions. This sensitivity allows chaos to occur in deterministic systems, which is the subject of much current research.

One way to experience sensitivity to changes in the initial conditions at first hand is to try a little "target practice." For the ODEs in Problems 24–27, use `dfield5` to find approximately the value of x_0 such that the solution $x(t)$ to the initial value problem with initial condition $x(0) = x_0$ satisfies $x(t_1) = x_1$. You should use the Keyboard input window to initiate the solution. After an unsuccessful attempt try again with another initial condition. The Uniqueness Theorem should help you limit your choices. If you make sure that the Display Window is the current figure (by clicking on it), and execute `plot(t1,x1,'or')` at the command line, you will have a nice target to shoot at.

You will find that hitting the target gets more difficult in each of these problems.

24. $x' = x - \sin(x)$, $t_1 = 5$, $x_1 = 2$.
25. $x' = x^2 - t$, $t_1 = 4$, $x_1 = 0$.
26. $x' = x(1 - x^2)$, $t_1 = 5$, $x_1 = 0.5$.
27. $x' = x\sin(x) + t$, $t_1 = 5$, $x_1 = 0$.