Exercise(s) 23. 4. 2026

The last exercise is to be handed in at the beginning of the next lecture.

21. For given constants $\gamma > 0$ and $\tau > 2$ the Diophantine conditions on frequency vectors are

$$\bigwedge_{k \in \mathbb{Z}^n \setminus \{0\}} \quad |2\pi \langle k \mid \alpha \rangle| \;\geq\; \frac{\gamma}{|k|^\tau} \;.$$

Show that the set $\Gamma_{\gamma,\tau} \subseteq \mathbb{R}^n$ of such Diophantine frequency vectors is nowhere dense and has relative Lebesgue measure tending to 1 as $\gamma \to 0$.

22. The aim of this exercise is to prove a Poincaré–Birkhoff fixed point theorem. Consider the annulus $A := \mathbb{S} \times [1, 2]$, with coordinates (x, y), where x is counted mod 2π. Consider a smooth, boundary preserving diffeomorphism $T_\varepsilon : A \longrightarrow A$, of the form $T_\varepsilon : (x, y) \mapsto (x + 2\pi\rho(y), y) + \varepsilon\; (f(x, y, \varepsilon), g(x, y, \varepsilon))$ and such that

- $\rho'(y) \neq 0$, saying that T_ε is a twist-map (for simplicitiy we take ρ increasing);
- $\oint_\gamma y \, \mathrm{d}x = \oint_{T_\varepsilon(\gamma)} y \, \mathrm{d}x$, which means that T_ε is preserving area.

Show that for each rational number p/q, with

$$\rho(1) \leq \frac{p}{q} \leq \rho(2) \;,$$

in A there exists a periodic point of T_ε, of period q, provided that $|\varepsilon|$ is sufficiently small. *Hint:* Abbreviating $T_\varepsilon^q(x, y) = (\Phi_{q,\varepsilon}(\varepsilon, y), y + O(\varepsilon))$, with $\Phi_{q,\varepsilon}(x, y) = x + 2\pi q\rho(y) + O(\varepsilon)$, consider the equation $\Phi_{q,\varepsilon}(x, y) = x + 2\pi p$, for $p \in \mathbb{Z}$. Use the implicit function theorem in order to obtain a curve $C = \{y = F(x, \varepsilon)\}$ of solutions. Then study the intersection of C and $T_\varepsilon^q(C)$.