

The last exercise is to be handed in at the beginning of the next lecture.

3. A result of this exercise turns out to be that

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_{n \times n} \mid \operatorname{tr} A = 0\}$$

is the Lie algebra of the Lie group

$$SL_n(\mathbb{R}) = \{S \in M_{n \times n} \mid \det S = 1\} .$$

To this end compute for a curve

$$\begin{aligned} \gamma : \mathbb{R} &\longrightarrow M_{n \times n} \\ t &\longmapsto \gamma(t) \end{aligned}$$

the corresponding tangent vector

$$\left. \frac{d}{dt} \gamma(t) \right|_{t=0}$$

and use the curve $\gamma(t) = \exp tA$ to conclude that $\mathfrak{sl}_n(\mathbb{R}) = T_{\operatorname{Id}} SL_n(\mathbb{R})$. What are the consequences for the flow of a divergence-free vector field?

4. A matrix $A \in M_{n \times n}(\mathbb{R})$ defines a linear differential equation $\dot{x} = Ax$, write \mathcal{L}_A for the corresponding vector field. Next to the commutator $[A, B] = AB - BA$ of two matrices there is also the Lie bracket $[\mathcal{L}_A, \mathcal{L}_B]$ defined by how vector fields act as derivations on smooth functions f , given as

$$[\mathcal{L}_A, \mathcal{L}_B](f) := \mathcal{L}_A(\mathcal{L}_B(f)) - \mathcal{L}_B(\mathcal{L}_A(f)) .$$

Show that

$$[\mathcal{L}_A, \mathcal{L}_B] = -\mathcal{L}_{[A, B]} .$$

Furthermore show that

$$\varphi_t \circ \psi_s - \psi_s \circ \varphi_t = st[A, B] + \mathcal{O}\left((s^2 + t^2)^{\frac{3}{2}}\right)$$

for the flows φ of \mathcal{L}_A and ψ of \mathcal{L}_B . Hence, $[\mathcal{L}_A, \mathcal{L}_B] = 0$ if these two flows commute.