Exercises 19. 2. 2026

The last exercise is to be handed in at the beginning of the next lecture.

5. Let $(V, +, 0, \cdot)$ be a vector space of finite dimension $\dim V = n$. Show that an associative unital algebra $(\Lambda, +, 0, \cdot, \wedge, 1)$ together with a linear mapping $\varphi : V \longrightarrow \Lambda$ are uniquely determined — up to canonical isomorphy — by the two properties

   (i) $\bigwedge_{v \in V} \quad \varphi(v) \wedge \varphi(v) \; = \; 0 \qquad (\text{so } \varphi(w) \wedge \varphi(v) = -\varphi(v) \wedge \varphi(w))$

   (ii) for every linear mapping $h : V \longrightarrow A$ into an associative unital algebra $(A, +, 0, \cdot, *, 1)$ satisfying

   $$\bigwedge_{v \in V} \quad h(v) * h(v) \; = \; 0$$

   there is a unique 1–preserving algebra-homomorphism $\hat{h} : \Lambda \longrightarrow A$ with $\hat{h} \circ \varphi = h$.

6. Denote by $\mathcal{P} = \mathcal{P}(\{1, \ldots, n\})$ the set of all subsets of $\{1, \ldots, n\}$ and by $\{e_T \mid T \in \mathcal{P}\}$ a fixed basis of $\mathbb{R}^{2^n}$. For $r, s \in \{1, \ldots, n\}$ put

   $$\sigma(r, s) \;\; := \;\; \begin{cases} 1 & r < s \\ 0 \quad \text{if} & r = s \\ -1 & r > s \end{cases}$$

   and

   $$\tau(R, S) \;\; := \;\; \prod_{r \in R} \prod_{s \in S} \sigma(r, s)$$

   for $R, S \in \mathcal{P}$. Show that

   $$e_R \; \wedge \; e_S \;\; := \;\; \tau(R, S) \cdot e_{R \cup S}$$

   turns $\mathbb{R}^{2^n}$ into an associative unital algebra $\Lambda(e_T \,|\, T \in \mathcal{P})$ with $e_\emptyset = 1$. Check that

   $$\bigwedge_{i, j \in \{1, \ldots, n\}} \quad e_{\{j\}} \; \wedge \; e_{\{i\}} \;\; = \;\; -e_{\{i\}} \; \wedge \; e_{\{j\}}$$

   and that

   $$e_T \;\; = \;\; e_{\{i_1\}} \; \wedge \; e_{\{i_2\}} \; \wedge \; \ldots \; \wedge \; e_{\{i_k\}}$$

   for all $T = \{i_1, \ldots, i_k\}$ with $i_1 < \ldots < i_k$. Let V be a vector space with basis $\{e_1, \ldots, e_n\}$. Prove that $\Lambda := \Lambda(e_T \,|\, T \in \mathcal{P})$ together with $\varphi : V \longrightarrow \Lambda$ defined by $\varphi(e_i) = e_{\{i\}}$, $i = 1, \ldots, n$ satisfy (i) and (ii) of the previous exercise.

   This unique algebra $\Lambda = \Lambda(V)$ is called the *exterior (or Graßmann) algebra* of V. Generalize from $\mathbb{R}$ to any field K. Can you also generalize to a ring R ? How important is the assumption of finite dimension?