Exercises 19. 2. 2026

The last exercise is to be handed in at the beginning of the next lecture.

5. Let $(V,+,0,\cdot)$ be a vector space of finite dimension $\dim V = n$. Show that an associative unital algebra $(\Lambda,+,0,\cdot,\wedge,1)$ together with a linear mapping $\varphi : V \longrightarrow \Lambda$ are uniquely determined — up to canonical isomorphy — by the two properties

   (i) $\displaystyle\bigwedge_{v\in V} \quad \varphi(v)\wedge\varphi(v) \;=\; 0 \qquad (\text{so } \varphi(w)\wedge\varphi(v) = -\varphi(v)\wedge\varphi(w))$

   (ii) for every linear mapping $h : V \longrightarrow A$ into an associative unital algebra $(A,+,0,\cdot,*,1)$ satisfying
   $$\bigwedge_{v\in V} \quad h(v) * h(v) \;=\; 0$$
   there is a unique 1–preserving algebra-homomorphism $\hat{h} : \Lambda \longrightarrow A$ with $\hat{h}\circ\varphi = h$.

6. Denote by $\mathcal{P} = \mathcal{P}(\{1,\ldots,n\})$ the set of all subsets of $\{1,\ldots,n\}$ and by $\{e_T \mid T \in \mathcal{P}\}$ a fixed basis of $\mathbb{R}^{2^n}$. For $r,s\in\{1,\ldots,n\}$ put
   $$\sigma(r,s) \;:=\; \begin{cases} 1 & r<s \\ 0 \quad \text{if} & r=s \\ -1 & r>s \end{cases}$$
   and
   $$\tau(R,S) \;:=\; \prod_{r\in R}\prod_{s\in S}\sigma(r,s)$$
   for $R,S\in\mathcal{P}$. Show that
   $$e_R \;\wedge\; e_S \;:=\; \tau(R,S)\cdot e_{R\cup S}$$
   turns $\mathbb{R}^{2^n}$ into an associative unital algebra $\Lambda(e_T\,|\,T\in\mathcal{P})$ with $e_\emptyset = 1$. Check that
   $$\bigwedge_{i,j\in\{1,\ldots,n\}} \quad e_{\{j\}} \;\wedge\; e_{\{i\}} \;=\; -e_{\{i\}} \;\wedge\; e_{\{j\}}$$
   and that
   $$e_T \;=\; e_{\{i_1\}} \;\wedge\; e_{\{i_2\}} \;\wedge\; \ldots \;\wedge\; e_{\{i_k\}}$$
   for all $T = \{i_1,\ldots,i_k\}$ with $i_1<\ldots<i_k$. Let V be a vector space with basis $\{e_1,\ldots,e_n\}$. Prove that $\Lambda := \Lambda(e_T\,|\,T\in\mathcal{P})$ together with $\varphi : V \longrightarrow \Lambda$ defined by $\varphi(e_i) = e_{\{i\}}$, $i=1,\ldots,n$ satisfy (i) and (ii) of the previous exercise.

   This unique algebra $\Lambda = \Lambda(V)$ is called the *exterior (or Graßmann) algebra* of V. Generalize from $\mathbb{R}$ to any field K. Can you also generalize to a ring R ? How important is the assumption of finite dimension?