

The last exercise is to be handed in at the beginning of the next lecture.

7. In this exercise we study some properties of the Poisson bracket on a symplectic manifold.

Show that if two functions F, G are integrals of H then so is $\{F, G\}$.

Show that if $h : \mathcal{P} \rightarrow \mathcal{P}$ is symplectic then $\{F, G\} \circ h = \{F \circ h, G \circ h\}$. What is the meaning of this equation?

8. Let Λ be the exterior algebra of an n -dimensional vector space V with basis $\{e_1, \dots, e_n\}$.

(i) Explain that the $e_{i_1} \wedge \dots \wedge e_{i_k}$, $i_1 < \dots < i_k$ together with 1 form a basis of Λ .

(ii) Check that for all subsets $\{i_1 < \dots < i_k\}, \{j_1 < \dots < j_\ell\} \subseteq \{1, \dots, n\}$ one has

$$e_{j_1} \wedge \dots \wedge e_{j_\ell} \wedge e_{i_1} \wedge \dots \wedge e_{i_k} = (-1)^{k\ell} e_{i_1} \wedge \dots \wedge e_{i_k} \wedge e_{j_1} \wedge \dots \wedge e_{j_\ell} .$$

(iii) Show that $\{v_1, \dots, v_m\} \subseteq V$ is linear dependent if and only if $v_1 \wedge \dots \wedge v_m = 0$.

(iv) Let $\{u_1, \dots, u_m\}, \{v_1, \dots, v_m\} \subseteq V$ with $\{u_1, \dots, u_m\}$ linear independent. Prove that $\langle u_1, \dots, u_m \rangle = \langle v_1, \dots, v_m \rangle$ have the same linear span if and only if $u_1 \wedge \dots \wedge u_m$ is a scalar multiple of $v_1 \wedge \dots \wedge v_m$.

Define $\Lambda^k V := \langle v_1 \wedge \dots \wedge v_k \mid v_i \in V \rangle$, the k th exterior power of V .

(v) Show that $\Lambda^k V = 0$ for $k > n$ and that for $k \leq n$ the $e_{i_1} \wedge \dots \wedge e_{i_k}$, $i_1 < \dots < i_k$ with the now fixed index k form a basis of $\Lambda^k V$. Conclude $\dim \Lambda^k V = \binom{n}{k}$. Identify $V = \Lambda^1 V$ and check

$$\Lambda = \bigoplus_{k=0}^n \Lambda^k V$$

and

$$\bigwedge_{k, \ell \in \mathbb{N}_0} \Lambda^k V \wedge \Lambda^\ell V = \Lambda^{k+\ell} V$$

where $A \wedge B := \langle a \wedge b \mid a \in A, b \in B \rangle$.

(vi) For $\varphi : V \rightarrow \Lambda(V)$ and $\psi : W \rightarrow \Lambda(W)$ show that for every $f \in L(V, W)$ there is a unique 1-preserving algebra-homomorphism $\Lambda(f) : \Lambda(V) \rightarrow \Lambda(W)$ satisfying $\Lambda(f) \circ \varphi = \psi \circ f$.

(vii) Prove that $\Lambda(f \circ g) = \Lambda(f) \circ \Lambda(g)$ if $g \in L(U, V)$ for some other vector space U with exterior algebra $\Lambda(U)$.

(viii) Check that $\Lambda(f)(\Lambda^k V) \subseteq \Lambda^k W$, yielding $\Lambda^k f \in L(\Lambda^k V, \Lambda^k W)$ and the splitting $\Lambda(f) = \bigoplus \Lambda^k(f)$. How is $\Lambda^n(f)$ for $f \in L(V)$ related to $\det(f)$?

(ix) Explain why $v_1 \wedge \dots \wedge v_m$ can be interpreted as the m -dimensional parallelepiped with sides v_1, \dots, v_m .