Exercises 26. 2. 2026

The last exercise is to be handed in at the beginning of the next lecture.

7. In this exercise we study some properties of the Poisson bracket on a symplectic manifold.

   Show that if two functions F, G are integrals of H then so is $\{F, G\}$.

   Show that if $h : \mathcal{P} \longrightarrow \mathcal{P}$ is symplectic then $\{F, G\} \circ h = \{F \circ h, G \circ h\}$. What is the meaning of this equation?

8. Let Λ be the exterior algebra of an n–dimensional vector space V with basis $\{e_1, \ldots, e_n\}$.

   (i) Explain that the $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $i_1 < \ldots < i_k$ together with 1 form a basis of Λ.

   (ii) Check that for all subsets $\{i_1 < \ldots < i_k\}, \{j_1 < \ldots < j_\ell\} \subseteq \{1, \ldots, n\}$ one has
   $$e_{j_1} \wedge \ldots \wedge e_{j_\ell} \ \wedge \ e_{i_1} \wedge \ldots \wedge e_{i_k} \quad = \quad (-1)^{kl} e_{i_1} \wedge \ldots \wedge e_{i_k} \ \wedge \ e_{j_1} \wedge \ldots \wedge e_{j_\ell} \ .$$

   (iii) Show that $\{v_1, \ldots, v_m\} \subseteq V$ is linear dependent if and only if $v_1 \wedge \ldots \wedge v_m = 0$.

   (iv) Let $\{u_1, \ldots, u_m\}, \{v_1, \ldots, v_m\} \subseteq V$ with $\{u_1, \ldots, u_m\}$ linear independent. Prove that $\langle u_1, \ldots, u_m \rangle = \langle v_1, \ldots, v_m \rangle$ have the same linear span if and only if $u_1 \wedge \ldots \wedge u_m$ is a scalar multiple of $v_1 \wedge \ldots \wedge v_m$.

   Define $\Lambda^k V := \langle v_1 \wedge \ldots \wedge v_k \mid v_i \in V \rangle$, the kth exterior power of V.

   (v) Show that $\Lambda^k V = 0$ for $k > n$ and that for $k \leq n$ the $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $i_1 < \ldots < i_k$ with the now fixed index k form a basis of $\Lambda^k V$. Conclude $\dim \Lambda^k V = \binom{n}{k}$. Identify $V = \Lambda^1 V$ and check
   $$\Lambda \quad = \quad \bigoplus_{k=0}^{n} \Lambda^k V$$
   and
   $$\bigwedge_{k,\ell \in \mathbb{N}_0} \quad \Lambda^k V \ \wedge \ \Lambda^\ell V \quad = \quad \Lambda^{k+\ell} V$$
   where $A \wedge B := \langle a \wedge b \mid a \in A, b \in B \rangle$.

   (vi) For $\varphi : V \longrightarrow \Lambda(V)$ and $\psi : W \longrightarrow \Lambda(W)$ show that for every $f \in L(V, W)$ there is a unique 1–preserving algebra-homomorphism $\Lambda(f) : \Lambda(V) \longrightarrow \Lambda(W)$ satisfying $\Lambda(f) \circ \varphi = \psi \circ f$.

   (vii) Prove that $\Lambda(f \circ g) = \Lambda(f) \circ \Lambda(g)$ if $g \in L(U, V)$ for some other vector space U with exterior algebra $\Lambda(U)$.

   $(viii)$ Check that $\Lambda(f)(\Lambda^k V) \subseteq \Lambda^k W$, yielding $\Lambda^k f \in L(\Lambda^k V, \Lambda^k W)$ and the splitting $\Lambda(f) = \bigoplus \Lambda^k(f)$. How is $\Lambda^n(f)$ for $f \in L(V)$ related to $\det(f)$?

   (ix) Explain why $v_1 \wedge \ldots \wedge v_m$ can be interpreted as the m–dimensional parallelepiped with sides $v_1, \ldots, v_m$.