Exercise 19. 3. 2026

The last exercise is to be handed in at the beginning of the next lecture.

13. Let $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function given by $H(x, y) = \frac{1}{2}y^2 + V(x)$ and assume that for $h \in \mathbb{R}$ the motion in the level set $H^{-1}(h)$ is periodic. Show that for energies near h the motion is also periodic. Let $A(h)$ denote the area enclosed by the level set $H^{-1}(h)$ and let $T(h)$ denote the period of the motion in this level set. Show that

$$T(h) \quad = \quad \left. \frac{\mathrm{d}A(z)}{\mathrm{d}z} \right|_{z=h}$$

and exemplify this for the harmonic oscillator with $V(x) = \frac{1}{2}x^2$. To this end, plot the level sets

$$H^{-1}(h) \quad = \quad \{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 2h \,\}$$

for $h > 0$ and compute the area $A(h)$ of the region enclosed by this level set. Then compare the period with the rate of change of the area.

14. Study the geodesic flow on a Lie group G with respect to a bi-invariant metric. Show that $G \times G$ is a symmetry group, compute the momentum mapping and explain the details of symmetry reduction. Use the reduced system to obtain as much information on the geodesic flow as you can. Consider in particular the case $G = SO(3)$ of a free rigid body and give an interpretation of your results.