Exercise(s) 9. 4. 2026

The last exercise is to be handed in at the beginning of the next lecture.

17. Let $\mathfrak{gl}(n,\mathbb{R})$ be the set of all real $n \times n$–matrices. Further define

$$\begin{aligned} Gl(n,\mathbb{R}) &= \{S \in \mathfrak{gl}(n,\mathbb{R}) \mid \det S \neq 0\} \\ O(n,\mathbb{R}) &= \{S \in \mathfrak{gl}(n,\mathbb{R}) \mid S^T S = \mathrm{id}\} \\ \mathfrak{o}(n,\mathbb{R}) &= \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T = -A\} \\ Sym(n,\mathbb{R}) &= \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T = A\}. \end{aligned}$$

    (a) Show that $\mathfrak{o}(n,\mathbb{R})$ and $Sym(n,\mathbb{R})$ are real vector spaces. Give their dimensions.

    (b) Show that $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{o}(n,\mathbb{R})$ form Lie algebras with respect to the product given by the commutator.

    (c) Show that $Gl(n,\mathbb{R})$ is an n^2–dimensional manifold. Show how $\mathfrak{gl}(n,\mathbb{R})$ can be regarded as the tangent space $T_{\mathrm{id}}Gl(n,\mathbb{R})$.

    (d) Let $F : \mathfrak{gl}(n,\mathbb{R}) \longrightarrow \mathfrak{gl}(n,\mathbb{R})$ be defined by $F(S) = S^T S$. Show that F is a smooth mapping and that the image of F is a subset of $Sym(n,\mathbb{R})$.

    (e) Show that the derivative[1] $D_{\mathrm{id}}F : \mathfrak{gl}(n,\mathbb{R}) \longrightarrow \mathfrak{gl}(n,\mathbb{R})$ is given by $D_{\mathrm{id}}F(B) = B^T + B$. What is the rank of this derivative?

    (f) Show that the rank of the derivative $D_S F : \mathfrak{gl}(n,\mathbb{R}) \longrightarrow \mathfrak{gl}(n,\mathbb{R})$ is independent of $S \in O(n,\mathbb{R})$. *Hint:* Use the fact that $O(n,\mathbb{R})$ is a group.

    (g) Show that $O(n,\mathbb{R})$ is a manifold and determine its dimension. In what sense can $\mathfrak{o}(m,\mathbb{R})$ be regarded as the tangent space $T_{\mathrm{id}}O(n,\mathbb{R})$?

18. The phase space $T^*SO(3)$ of the Lagrange top has the left trivialization $SO(3) \times \mathfrak{so}(3)$ to body co-ordinates (g,ℓ) and the right trivialization $\mathfrak{so}(3) \times SO(3)$ to spatial co-ordinates (μ, g). Let μ_3 denote the component of the angular momentum along the gravitiy axis and ℓ_3 denote the component of the angular momentum along the figure axis of the body. Show that $\{\mu_3, \ell_3\} = 0$.

[1] In another notation, $D_{\mathrm{id}}F = F_{*,\mathrm{id}}$.