Exercise(s) 12. 2. 2015

1. A matrix $A \in M_{n\times n}(\mathbb{R})$ defines a linear differential equation $\dot{x} = Ax$, write $\mathcal{L}_A$ for the corresponding vector field. Next to the commutator $[A, B] = AB - BA$ of two matrices there is also the Lie bracket $[\mathcal{L}_A, \mathcal{L}_B]$ defined by how vector fields act as derivations on smooth functions f, given as

$$[\mathcal{L}_A, \mathcal{L}_B](f) \;\; := \;\; \mathcal{L}_A(\mathcal{L}_B(f)) \; - \; \mathcal{L}_B(\mathcal{L}_A(f)) \;\; .$$

Show that

$$[\mathcal{L}_A, \mathcal{L}_B] \;\; = \;\; -\mathcal{L}_{[A,B]} \;\; .$$

Furthermore show that

$$\varphi_t \circ \psi_s \; - \; \psi_s \circ \varphi_t \;\; = \;\; s\,t\,[A, B] \; + \; \mathcal{O}\left((s^2 + t^2)^{\frac{3}{2}}\right)$$

for the flows φ of $\mathcal{L}_A$ and ψ of $\mathcal{L}_B$. Hence, $[\mathcal{L}_A, \mathcal{L}_B] = 0$ if these two flows commute.

2. A result of this exercise turns our to be that

$$\mathfrak{sl}_n(\mathbb{R}) \;\; = \;\; \{A \in M_{n\times n} \mid \operatorname{tr} A = 0\}$$

is the Lie algebra of the Lie group

$$SL_n(\mathbb{R}) \;\; = \;\; \{S \in M_{n\times n} \mid \det S = 1\} \;\; .$$

To this end compute for a curve

$$\begin{array}{rcccc} \gamma & : & \mathbb{R} & \longrightarrow & M_{n\times n} \\ & & t & \longmapsto & \gamma(t) \end{array}$$

the corresponding tangent vector

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right|_{t=0}$$

and use the curve $\gamma(t) = \exp tA$ to conclude that $\mathfrak{sl}_n(\mathbb{R}) = T_{\mathrm{Id}} SL_n(\mathbb{R})$. What are the consequences for the flow of a divergence-free vector field?