Exercise(s) 26. 2. 2015

1. Let's give this exercise a second chance.

   Let $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a given smooth function, with corresponding Hamiltonian vector field X_H. Here we use the standard symplectic structure on $\mathbb{R}^2$. Moreover, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism. Consider both the function $K := H \circ g^{-1}$, together with the associated Hamiltonian vector field X_K, and the transformed vector field $g_\star(X_H)$, defined by $g_\star(X_H)(g(p)) := D_p g X_H(p)$. Show that

   $$g_\star(X_H) \quad = \quad \det(Dg) \cdot X_K \ .$$

   Hint: exploit a coordinate free formulation of the fact that X_H is the Hamiltonian vector field corresponding to H. Discuss the implication for the integral curves of $g_\star(X_H)$ and X_K. Also consider the time-parametrisation of these curves. What happens in the special case that g is canonical?

2. Prove "Cartan's magic formula"

   $$\mathcal{L}_X \alpha \quad = \quad \mathrm{d}(\iota_X \alpha) \ + \ \iota_X \mathrm{d}\alpha$$

   where $\mathcal{L}_X \alpha$ is the Lie derivative of the differential form α along the vector field X and $\iota_X \alpha$ is the interior product of X with the k–form α, resulting in the $(k-1)$–form

   $$\iota_X \alpha(Y_2, \ldots, Y_k) \quad = \quad \alpha(X, Y_2, \ldots, Y_k) \ .$$