Exercise 21. 5. 2015

1. Consider the parameter-dependent Hamiltonian function

$$H(x, y, q, p; \omega, \lambda) \;=\; y_1 \;+\; \omega y_2 \;+\; \frac{p^2}{2} \;+\; \frac{q^4}{24} \;+\; \lambda \frac{q^2}{2}$$

on $\mathbb{T}^2 \times \mathbb{R}^4$ and determine for which $(q_0, p_0) \in \mathbb{R}^2$ there are invariant tori $\mathbb{T}^2 \times \{(y_1, y_2, q_0, p_0)\}$. In the elliptic case (with normal frequency α) such tori are known to persist a small perturbation if the Diophantine conditions

$$\bigwedge_{\substack{k \in \mathbb{Z}^2 \\ k \neq 0}} \; \bigwedge_{\substack{\ell \in \mathbb{Z} \\ |\ell| \leq 2}} \quad |2\pi(k_1 1 + k_2 \omega) \;+\; \ell \alpha| \;\geq\; \frac{\gamma}{|k|^\tau} \;. \tag{1}$$

are satisfied, where $\gamma > 0$ and $\tau > 1$, while for hyperbolic tori the Diophantine conditions (1) with $\ell = 0$ on the internal frequencies suffice. Sketch in the (ω, λ)–plane where the necessary conditions for the 2–tori at the origin $(q_0, p_0) = (0, 0)$ are satisfied. What can you say about the 2–tori that bifurcate off from the origin?