Geometric Mechanics

This hand-in exercise is due on 23 March.

A). Consider a Hamiltonian system defined on $\mathbb{R}^6$ with an equilibrium point at the origin for which the Taylor expansion reads as $H(x,y) = H_0^0(x,y) + H_1^0(x,y) + \dots$ with

$$H_0^0(x,y) \quad = \quad \frac{y_1^2 + x_1^2}{2} \;+\; y_2^2 \;+\; x_2^2 \;+\; y_3^2 \;+\; x_3^2 \;.$$

Note that the equilibrium is elliptic with frequencies in 1:2:2 resonance.

1. Compute the flow of the vector field defined by H_0^0.
2. Give the spectrum of the linear mapping $X_{H_0^0} : \mathcal{G}_{k+2} \longrightarrow \mathcal{G}_{k+2}$ for general $k \in \mathbb{N}$.
3. Specify this for $k = 1$ and determine complex polynomial bases for the subspaces in the splitting $\mathcal{G}_3 = \ker X_{H_0^0} \oplus \operatorname{im} X_{H_0^0}$.
4. Determine from this real polynomial bases for these subspaces.
5. Show that the third order normal form of H can be brought into the form

$$H(q,p) \quad = \quad H_0^0(q,p) + A\left(\frac{p_1^2 - q_1^2}{2}\, q_2 \;-\; p_1 q_1 p_2\right) + B\left(\frac{p_1^2 - q_1^2}{2}\, q_3 \;-\; p_1 q_1 p_3\right) + \dots$$

   by means of a Poisson transformation $(x,y) \mapsto (q,p)$. *Hint:* use rotations in the (q_2,p_2)–plane and in the (q_3,p_3)–plane to get rid of the terms

$$\frac{p_1^2 - q_1^2}{2}\, p_2 \;+\; p_1 q_1 q_2 \quad \text{and} \quad \frac{p_1^2 - q_1^2}{2}\, p_3 \;+\; p_1 q_1 q_3 \;.$$

6. Use a rotation in the 4–dimensional (q_2,p_2,q_3,p_3)–space to achieve $B = 0$. Conclude that the truncated third order normal form $H_0^0(q,p) + H_0^1(q,p)$ around an elliptic equilibrium with frequencies in 1:2:2 resonance is integrable, having three independent integrals of motion.

B). Consider a Hamiltonian system defined on $\mathbb{R}^6$ with an equilibrium point at the origin for which the Taylor expansion reads as $H(x,y) = H_0^0(x,y) + H_1^0(x,y) + \dots$ with

$$H_0^0(x,y) \quad = \quad -\frac{y_1^2 + x_1^2}{2} \;+\; y_2^2 \;+\; x_2^2 \;+\; y_3^2 \;+\; x_3^2 \;.$$

Note that the equilibrium is elliptic, but not an extremum of the Hamiltonian function. One speaks of an equilibrium in -1:2:2 resonance. Show that the truncated third order normal form is integrable.