Geometric Mechanics 2009

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Exercise Sheet 1

Action variables and periods of 1-degree-of-freedom systems.

Return by Monday, 20$^{\text{th}}$ April

1. Consider the harmonic oscillator with Hamilton function

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

on the phase space $(p,q) \in \mathbb{R}^2$.

(a) For $E > 0$, plot the level sets

$$M_E = \{(p,q) \in \mathbb{R}^2 \,:\, H(p,q) = E\} \tag{1}$$

and compute the area, $A(E)$, of the region in the (p,q) plane enclosed by the level set M_E.

(b) Show that

$$T(E) := \frac{\mathrm{d}A(E)}{\mathrm{d}E}\,,$$

is the time (the period) which it takes the harmonic oscillator at energy E to trace the full level set M_E exactly one time.

2. Consider the Morse oscillator described by the Hamilton function

$$H = \frac{p^2}{2m} + D_e\big(\mathrm{e}^{-2aq} - 2\mathrm{e}^{-aq}\big)\,,$$

where D_e and a are positive constants, and $(p,q) \in \mathbb{R}^2$.

(a) A Morse oscillator is often used to describe a chemical bond. What is the meaning of D_e in this case?

(b) Show that there are two critical energies, $E_1 < E_2$, such that the level sets M_E defined as in (1) are empty if $E < E_1$, topological circles if $E_1 < E < E_2$, and topological lines for $E > E_2$. Plot the level sets for the energies E_1, E_2, an energy between E_1 and E_2, and an energy greater than E_2.

(c) For $E_1 < E < E_2$, compute the area, $A(E)$, of the region enclosed by the level set M_E in the (p,q) plane. For such energies, compute the period

$$T(E) = \frac{\mathrm{d}A(E)}{\mathrm{d}E}\,.$$

3. Consider the pendulum with Hamilton function

$$H = \frac{p^2}{2m} - mgl\cos q \quad (g,l > 0)$$

with $(p,q) \in \mathbb{R}\times S^1 := \mathbb{R}\times\big(\mathbb{R}/(2\pi\mathbb{Z})\big)$.

(a) Show that there are two critical energies, $E_1 < E_2$, such that the level set

$$M_E = \{(p, q) \in \mathbb{R} \times \big(\mathbb{R}/(2\pi\mathbb{Z})\big) \;:\; H(p, q) = E\}$$

is empty if $E < E_1$, consists of one topological circle if $E_1 < E < E_2$, and two topological circles if $E > E_2$. Plot the level sets M_E for the energies E_1, E_2, an energy between E_1 and E_2, and an energy greater than E_2.

(b) For $E_1 < E < E_2$ compute the area, $A_{\rm osc}(E)$, enclosed by the level set M_E. For $E > E_2$, compute the area, $A_{\rm rot}(E)$ enclosed by either of the lines in the level set M_E and the axis $p = 0$. Note that the integrals involved in the computation of $A_{\rm osc}(E)$ and $A_{\rm rot}(E)$ are elliptic integrals which you can look up in an integral table (see, e.g., the book *Handbook of Mathematical Functions* by Abramowitz and Stegun which is available online under the URL www.math.hkbu.edu.hk/support/aands/toc.htm). Use a program like Maple or Mathematica to plot $A_{\rm osc/rot}(E)$ versus E.

(c) Determine the periods $T_{\rm osc/rot}(E) = \mathrm{d}A_{\rm osc/rot}(E)/\mathrm{d}E$. Again use a program like Maple or Mathematica to plot $T_{\rm osc/rot}(E)$ versus E. Use the asymptotics of elliptic integrals to show that for $E \to E_1$, $A_{\rm osc}(E)$ behaves like the corresponding function of an harmonic oscillator. Similarly study $A_{\rm rot}(E)$ for $E \to \infty$.