**Geometric Mechanics** 2009

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## Exercise Sheet 2

## Hamilton-Jacobi Equation.

Note: this problem sheet will *not* be graded

1. Consider the motion of a particle under a gravitational force described by the Hamilton function

$$H = \frac{p_x^2 + p_y^2}{2m} + mgy$$

on the phase space $(p_x, p_y, x, y) \in \mathbb{R}^4$. Find a complete solution of the Hamilton-Jacobi equation for this problem.

2. Consider the motion of a particle on a 2-dimensional sphere $x^2 + y^2 + z^2 = 1$ with unit radius under the influence of a gravitational force acting in the negative $z$ direction. Write down the Hamilton function in spherical coordinates, and show that the corresponding Hamilton-Jacobi equation is completely separable.

3. Consider the motion of a particle in the three dimensional space $\mathbb{R}^3$ under the influence of two space fixed attracting centers. In a suitable scaling and Cartesian coordinates the corresponding Hamiltonian reads

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2} - \frac{\mu}{r_1} - \frac{1-\mu}{r_2}\,.$$

Here $r_1^2 = (x+1)^2 + y^2 + z^2$ and $r_2^2 = (x-1)^2 + y^2 + z^2$ are the squared distances of the particle to the two centers located on the $x$ axis at $x = \pm 1$, and $\mu \in (0, 1)$ is a fixed parameter. Let $\xi = (r_1 + r_2)/2$ and $\eta = (r_1 - r_2)/2$, and $\phi$ denote the angle of the particle about the $x$ axis. These are so-called *elliptic coordinates* in terms of which the Hamilton function $H$ assumes the form

$$H = \frac{1}{2} p_\xi^2 \frac{\xi^2 - 1}{\xi^2 - \eta^2} + \frac{1}{2} p_\eta^2 \frac{1 - \eta^2}{\xi^2 - \eta^2} + \frac{1}{2} \frac{p_\phi^2}{\xi^2 - \eta^2} \left( \frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2} \right) - \frac{\xi}{\xi^2 - \eta^2} - \frac{(1 - 2\mu)\eta}{\xi^2 - \eta^2}\,.$$

Show that the resulting Hamilton-Jacobi equation completely separates with respect to the ellipitic coordinates.