GEOMETRIC MECHANICS: 2009

ARTHEMY V. KISELEV

Deadline May 25, 2009

1. Let $\pi\colon \mathbb{R}^{m+n} \to \mathbb{R}^m$ be a trivial fibre bundle. Find $\dim J^k(\pi) = {?}$

2. Verify that $\varphi = f(x)\cdot u_x + \frac{1}{2}f'(x) + g(y)\cdot u_y + \frac{1}{2}g'(y)$ is a symmetry of the Liouville equation $\mathcal{E}_{\text{Liou}} = \bigl\{u_{xy} = \exp(2u)\bigr\}$ for any f, g. Find φ-invariant solutions of $\mathcal{E}_{\text{Liou}}$.

3. Find any one-parametric family of travelling-wave solutions for the Korteweg–de Vries equation $u_t = -u_{xxx} + 6uu_x$.

4. Consider the extension $\mathcal{E}(\epsilon) = \bigl\{\tilde{u}_t = -\frac{1}{2}\tilde{u}_{xxx} + 3\tilde{u}\tilde{u}_x + 3\epsilon^2\tilde{u}^2\tilde{u}_x\bigr\}$ of the equation $\mathcal{E}(0) = \bigl\{u_t = -\frac{1}{2}u_{xxx} + 3uu_x\bigr\}$.
   - Verify that Gardner's map $\mathfrak{m}_\epsilon = \bigl\{u = \tilde{u} \pm \epsilon\tilde{u}_x + \epsilon^2\tilde{u}^2\bigr\}$ yields the solution $u(x,t)$ of $\mathcal{E}(0)$ for any solution $\tilde{u}(x,t;\epsilon)$ of $\mathcal{E}(\epsilon)$.
   - Show that all the Taylor coefficients $\tilde{u}_k(x,t)$ in the expansion $\tilde{u} = \sum\limits_{k=0}^{+\infty} \tilde{u}_k(x,t)\cdot\epsilon^k$ are conserved densities for $\mathcal{E}(0)$, and derive the recurrence relation between them.
   - Prove that all such odd-index densities are trivial: $\tilde{u}_{2k+1} = D_x(\cdots)$.

5. Prove Ibragimov's identity

$$\mathcal{E}_\varphi = \varphi\cdot\mathbf{E}_u + \sum_{i=1}^{n} D_{x^i}\circ Q_{\varphi,i},$$

where

$$Q_{\varphi,i} = \sum_{\tau}\sum_{\rho+\eta=\tau} (-1)^{|\eta|}\cdot D_\rho(\varphi)\cdot D_\eta\circ\frac{\partial}{\partial u_{\tau+1_i}}.$$