8

Area preserving mappings

In one degree of freedom one obtains 1–dimensional curves when fixing the Hamiltonian $H = h$ and these are[1] the trajectories of the vector field X_H. In this way the complete solution of the equations of motion is brought back to an integral like (2.2) to determine the time-parametrisation of the orbits. In this sense every Hamiltonian system with one degree of freedom is an integrable system.

In two degrees of freedom the energy shells $\{H = h\}$ are 3–dimensional and one would need two more constants of motion to single out 1–dimensional trajectories. In the previous sections we have seen how already one conserved quantity (independent of the Hamiltonian) allows to reduce to one degree of freedom. In addition to the solution $(q(t), p(t))$ in one degree of freedom we have to solve

$$\dot{\rho} \;=\; \frac{\partial H}{\partial \mu}(q(t), p(t); \mu) \tag{8.1}$$

where ρ is conjugate to the conserved quantity. Since the right hand side of (8.1) does not depend on ρ this amounts to one more integration; all two-degrees-of-freedom systems that admit an extra conserved quantity are integrable.

Thus, given a Hamiltonian dynamical system in two degrees of freedom it is rewarding to look for an extra conserved[2] quantity. The simplifying assumptions usually made when modelling a mechanical system often introduce additional symmetries. Consequently, some of the problems from classical mechanics, like the Kepler system or the spherical pendulum, turned out to be integrable. Eventually it became clear that integrable systems are the ex-

[1] Where the derivative (or gradient) of the Hamiltonian vanishes one obtains equilibria, and unstable equilibria form an energy level set together with their stable and unstable manifold(s).

[2] One also speaks of a second integral of motion.

ception and non-integrable systems are the rule; Poincaré[3] showed that the so-called circular restricted three body problem is not integrable.

One approach to non-integrable systems is to approximate by an integrable system, typically obtained by normalization. In fact, this can be thought of as making the simplifying assumptions introducing extra symmetry part of the mathematical theory. For a satisfactory description of the resulting integrable systems some kind of "robustness" is desirable – the perturbation from the normal form back to the original should not completely invalidate the description of the dynamics.

In one degree of freedom we could easily draw phase portraits of the 2–dimensional dynamics. The direct translation to two degrees of freedom would result in pictures of too high dimension. What is still possible is to fix the value of the energy and depict the flow on the resulting invariant manifold, at least locally. Similarly a 3–dimensional phase portrait is obtained when the Hamiltonian system depends only on two variables $q, p \in \mathbb{R}$ but is not autonomous, the Hamiltonian $H = H(t, q, p)$ depending explicitly on time. Then the resulting equations of motion are

$$\dot{t} = 1 \tag{8.2a}$$

$$\dot{q} = \frac{\partial H}{\partial p} \tag{8.2b}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{8.2c}$$

and define a flow $\phi(t, t_0, q_0, p_0)$ on $\mathbb{R}^3$; for autonomous systems one has $\varphi(t - t_0, q_0, p_0) = \phi(t, t_0, q_0, p_0)$ but in the non-autonomous case the initial time t_0 becomes important and cannot simply be put to $t_0 = 0$ by passing from t to $t - t_0$. One also speaks of a Hamiltonian system with $1\frac{1}{2}$ degrees of freedom, but in its complexity the dynamics is closer to 2 degrees of freedom than to 1 degree of freedom. In fact, defining

$$K(x, y) = y_1 + H(x_1, x_2, y_2)$$

one otains the equations of motion

$$\dot{x}_1 = \frac{\partial K}{\partial y_1} = 1 \tag{8.3a}$$

$$\dot{y}_1 = -\frac{\partial H}{\partial x_1}(x_1, x_2, y_2) \tag{8.3b}$$

$$\dot{x}_2 = \frac{\partial H}{\partial y_2}(x_1, x_2, y_2) \tag{8.3c}$$

$$\dot{y}_2 = -\frac{\partial H}{\partial x_2}(x_1, x_2, y_2) \tag{8.3d}$$

[3] *Acta Mathematica* **13**, p. 5–271 (1890) and *Les Méthodes Nouvelles de la Mécanique Céleste* (1892)–(1899).

that differ from (8.2) only by the extra equation (8.3b). Moreover, once the three "initial" equations (8.3a), (8.3c) and (8.3d) are solved the solution is completed by $y_1(t) = K(x(0), y(0)) - H((x_1(t), x_2(t), y_2(t))$. Still, the Hamiltonian system (8.3) is a bit more special than a general Hamiltonian system in two degrees of freedom as there are no equilibria.

An approach to (8.2) that has proven to be very efficient is to sample the orbit $(q(t), p(t)) = \phi(t, 0, q, p)$ at a sequence $t_k = k \in \mathbb{Z}$ of time values.

Theorem 8.1. The stroboscopic or time–1–mapping

$$F : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (q,p) & \mapsto & \phi(1,0,q,p) \end{array}$$

is area preserving.

Proof. Starting from a (small) disk $D \subset \mathbb{R}^2$ consider the volume $V \subset \mathbb{R}^3$ formed by all trajectories starting in D, with time varying in the interval $[0, 1]$. Thus, the boundary ∂V of V is given by D, $F(D)$ and the "cylinder" formed by the trajectories starting in ∂D. Applying Gauß' Theorem to the vector field Y on $\mathbb{R}^3$ defined by (8.3) yields

$$0 \;=\; \int_V \mathrm{div} Y \,\mathrm{d}t\mathrm{d}q\mathrm{d}p \;=\; \int_{\partial V} \langle Y \mid \mathrm{d}\sigma \rangle \;=\; \int_{F(D)} \mathrm{d}q\mathrm{d}p \;-\; \int_D \mathrm{d}q\mathrm{d}p$$

since Y is divergence-free and tangent to the cylinder $\partial V^{\setminus D \cup F(D)}$. Thus, $F(D)$ has the same area as D. □

Alternatively one can adjust the proof of Liouville's Theorem 4.7 to the present non-autonomous situation. The time–1–mapping has a stroboscopic nature only if the dependence of H on t is 1–periodic (in case of T–periodicity one works with the time–T–mapping or rescales time accordingly). Then $\phi(t+1, t_0+1, q, p) = \phi(t, t_0, q, p)$ and the kth iterate is given by $F^k(q, p) = \phi(k, 0, q, p)$. Since H is periodic in t we can divide out the $\mathbb{Z}$–action $(k, (t, q, p)) \mapsto (t + k, q, p)$ on $\mathbb{R}^3$ and pass to the quotient space $\mathbb{T} \times \mathbb{R}^2$. Most properties of the flow turn out to be governed by the discrete dynamical system defined by F. In this way properties of discrete dynamical systems on $\mathbb{R}^2$ lead to information on periodically varying Hamiltonian system in $1\frac{1}{2}$ degrees of freedom. Conversely, all properties of area preserving discrete dynamical systems can be derived from non-autonomous Hamiltonian systems.

Theorem 8.2. Given an invertible smooth area preserving mapping $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ there is a time-dependent Hamiltonian $H \in C^\infty(\mathbb{T} \times \mathbb{R}^2)$ for which F is the time–1–mapping.

Proof. Consider on $[0, 1] \times \mathbb{R}^2$ the constant vector field (8.2) with $H = 0$. We use the resulting flow $\phi(t, 0, q, p) = (t, q, p)$ to define a flow on $\mathbb{T} \times \mathbb{R}^2$ by glueing $\{0\} \times \mathbb{R}^2$ and $\{1\} \times \mathbb{R}^2$ together along the diffeomorphism F. The relation

$$(t,q,p) \sim (s,x,y) \quad :\Leftrightarrow \quad \begin{cases} \quad t = s,\ q = x,\ p = y \\ \text{or}\ \ t = 0,\ s = 1,\ (x,y) = F(q,p) \\ \text{or}\ \ t = 1,\ s = 0,\ (q,p) = F(x,y) \end{cases}$$

is an equivalence relation with quotient space

$$[0,1] \times \mathbb{R}^2 /_{\sim} \quad = \quad \mathbb{T} \times \mathbb{R}^2$$

and quotient flow

$$\phi(t,t_0,q,p) \quad = \quad \left(t, F^{\lfloor t \rfloor - \lfloor t_0 \rfloor}(q,p)\right)$$

made smooth. Here $\lfloor t \rfloor = \max\{k \in \mathbb{Z} \mid k \leq t\}$. □

The phase portraits of discrete dynamical systems give a fair impression of periodically forced Hamiltonian systems. According to [13] these all look more or less the same, see *e.g.* Fig. **?** for the dynamics defined by

$$\begin{pmatrix} q \\ p \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} f(q) \ - \ p \\ q \ - \ f(f(q) - p) \end{pmatrix} \tag{8.4}$$

with $f(q) = \eta q - (1-\eta)q^2, \eta = -5/4$ (other values of η yield similar phase portraits). Counterexamples are time–1–mappings of autonomous Hamiltonian systems (thus, with only one degree of freedom), which look more orderly, and ergodic area-preserving mappings, which look more chaotic.

Exercise 8.1. Choose an area-preserving mapping, *e.g.* the Hénon mapping

$$\begin{pmatrix} q \\ p \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 \ - \ \eta q^2 \ + \ p \\ -q \end{pmatrix}$$

or (8.4) for some function f and get a first impression of the dynamics by numerically drawing a phase portrait.

Exercise 8.2. Classify all discrete dynamical systems $\mathbb{Z} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that are defined by iterating a linear area preserving mapping $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Similar to the situation around equilibria in the continuous case, the linearization of F at a fixed point governs the dynamics near that fixed point for discrete dynamical systems. Before studying this more closely let us consider in how far area preserving mappings reflect the dynamics of general Hamiltonian systems in two degrees of freedom.

8.1 Poincaré-sections

The flow φ on a 3–dimensional energy shell $\{H = h\}$ in two degrees of freedom can be visualized by means of a Poincaré-section, a surface Σ_h that is transverse to the flow. Compute for every $z \in \Sigma_h$ the (first) return time

$$T(z) \ := \ \min \left\{ T > 0 \ \middle| \ \varphi_T(z) \in \Sigma_h \right\}$$

and define the Poincaré-mapping

$$\begin{array}{rcccl} F & : & \Sigma_h & \longrightarrow & \Sigma_h \\ & & z & \mapsto & \varphi_{T(z)}(z) \end{array}$$

(excluding from Σ_h those points that do not return, if necessary). The resulting phase portraits are again 2–dimensional. Note that not every aspect of the recurrent dynamics is captured in this way, for instance, a transverse section Σ_h cannot contain equilibria.

In general a Poincaré-section is a hypersurface transverse to the flow of a dynamical system, and the above definition of the Poincaré-mapping carries over verbatim. To emphasize that the energy has been fixed (whence Σ_h is of co-dimension 1 only within the 3–dimensional energy shell) we call Σ_h an iso-energetic Poincaré-section and F an iso-energetic Poincaré-mapping.

Again we want F to be area preserving. Emulating the proof of Darboux's Theorem 4.5 we can near Σ_h complete t and H by q and p to a canonical co-ordinate system. Hence (q,p) provides local co-ordinates on Σ_h. If $\Sigma_h = \{H = h, t = 0\}$ and $T(z) = 1$ for all $z \in \Sigma_h$ then also $\Sigma_h = \{H = h, t = 1\}$ and the proof of Theorem 8.1 still applies whence F is area preserving.

In fact, the area element $\mathrm{d}q\mathrm{d}p$ is preserved independently of the extra condition on Σ_h. Note that this area element is well-defined (independent of the particular co-ordinate system) since the transformation from (q,p) to (x,y) has Jacobi determinant $\det \dfrac{\partial(x,y)}{\partial(q,p)} \equiv 1$ if (t,H,x,y) form again a canonical co-ordinate system. In this way the iso-energetic Poincaré-mapping is area preserving. In general F does not preserve the area element inherited on Σ_h from the embedding in $\{H = h\}$; for co-ordinates

$$(q,p) \ \mapsto \ (t(q,p), h, q, p) \in \Sigma_h$$

the latter is given by

$$\sqrt{1 \ + \ (\frac{\partial t}{\partial q})^2 \ + \ (\frac{\partial t}{\partial p})^2} \, \mathrm{d}q\mathrm{d}p \ .$$

Working with $\mathrm{d}q\mathrm{d}p$ gives more flexibililty in the choice of Σ_h as only transversity to the flow has to be secured. In applications one often defines Σ by fixing some co-ordinate and then $\Sigma_h := \Sigma \cap \{H = h\}$.

Exercise 8.3. Define a Poincaré-section Σ for the spherical pendulum at $x_1 = 0$ and use $(x_3, y_3) \in]-1,1[\times \mathbb{R}$ as co-ordinates on Σ_h. Check that this fixes

$$x_2 \ = \ \sqrt{1 - x_3^2} \ , \quad y_2 \ = \ -\frac{x_3 y_3}{x_2} \quad \text{and} \quad y_1 \ = \ \sqrt{2h - 2\gamma x_3 - y_2^2 - y_3^2}$$

after a (which?) choice of signs. Show that $\frac{\mathrm{d}x_3 \mathrm{d}y_3}{1 - x_3^2}$ is the invariant area element and that for $h > -\gamma$ the iso-energetic Poincaré-mapping on Σ_h defined by the flow of the spherical pendulum coincides with the time–1–mapping of a one-degree-of-freedom Hamiltonian $H = H(x_3, y_3)$.

We call an area preserving mapping $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ integrable if there is a Hamiltonian function $H \in C^\infty(\mathbb{R}^2)$ with flow φ such that F coincides with the time–1–mapping $\varphi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

8.2 Fixed points

Fixed points of an area preserving mapping correspond to periodic orbits of the Hamiltonian system — of period 1 in the case of a stroboscopic mapping, in case of a Poincaré-mapping the period coincides with the orbit-dependent return time. Periodic points of an area preserving mapping also correspond to periodic orbits of the Hamiltonian system, but the period is larger and given by the sum of the return times ($1 + \ldots + 1$ for a stroboscopic mapping). For the local dynamics of a fixed point we may restrict to an open subset of $\mathbb{R}^2$ and move the fixed point to the origin. Then $F(0) = 0$, the constant term of F vanishes, and the linear part $DF(0)$ governs the dynamics near the fixed point. Area preservation implies $\det DF(0) = 1$ and the trace of $DF(0)$ allows to distinguish between the following cases.

$\operatorname{trace} DF(0) > 2$. The eigenvalues are $\lambda > 1$ and $0 < \frac{1}{\lambda} < 1$. The orbits of the linearization proceed along hyperbolas and the fixed point is called (directly) hyperbolic.

$\operatorname{trace} DF(0) = 2$. The linear part is equal to the identity or has Jordan normal form $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

$-2 < \operatorname{trace} DF(0) < 2$. The eigenvalues are $\mathrm{e}^{\pm \mathrm{i}\alpha}$ and the orbits proceed in rigid rotations of angle α along ellipses. The fixed point is called elliptic.

$\operatorname{trace} DF(0) = -2$. The linear part $DF(0) = -\mathrm{id}$ or has Jordan normal form $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$.

$\operatorname{trace} DF(0) < -2$. The eigenvalues are $\lambda < -1$ and $0 > \frac{1}{\lambda} > -1$. The orbits of the linearization jump between two hyperbolas and the fixed point is called (inversely) hyperbolic.

Hyperbolic fixed points are dynamically unstable, but structurally stable.

Theorem 8.3. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism with hyperbolic fixed point at the origin. Then there is a homeomorphism ψ that locally conjugates F to the linearization, satisfying $\psi \circ F = DF(0) \circ \psi$ on a small neighbourhood of the origin.

The proof furthermore shows that the local dynamics near a directly hyperbolic fixed point is conjugate to that of the linear system defined by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and near an inversely hyperbolic fixed point to that of $-A$. Since small perturbations of hyperbolic fixed points remain hyperbolic (with slightly perturbed eigenvalues), this shows (local) structural stability — the unperturbed and perturbed system have conjugate dynamics near the respective fixed points. While Theorem 8.3 remains true for hyperbolic fixed points in higher dimensions, the present 2–dimensional case allows to obtain a higher regularity for the conjugation ψ.

Theorem 8.4. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^2–diffeomorphism with hyperbolic fixed point at the origin. Then there is a C^1–diffeomorphism ψ that locally conjugates F to $DF(0)$.

Since $D\psi(0)$ conjugates the linear parts of the conjugated mappings, the eigenvalues are moduli that cannot be changed by means of differentiable conjugations. Note that both Theorems 8.3 and 8.4 do not require F to be area preserving.

In the elliptic case α is the rotation number on the invariant ellipses. This makes the eigenvalues not only smooth invariants as in the hyperbolic case but even topological invariants, preventing different linear parts to be conjugate to each other. The best possible simplification for the linear part is to choose (linear) co-ordinates in which

$$DF(0) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} .$$

Normal form theory aims to push the rotational symmetry generated by the linear part through the Taylor series. Starting point is the expansion

$$F(x, y) = F_1 + F_2 + \ldots$$

with linear term $F_1 = DF(0) \binom{x}{y}$ and

$$F_k = \begin{pmatrix} F_k^1 \\ F_k^2 \end{pmatrix} \in \mathcal{G}_k \times \mathcal{G}_k .$$

The search is for $\psi = \mathrm{id} + \ldots$ making $\psi \circ F \circ \psi^{-1}$ as simple as possible. To ensure that the resulting diffeomorphism is again area preserving we choose $\psi = \varphi_W = \varphi_1^W$ to be the time–1–mapping of a suitable Hamiltonian vector field X_W. This implies that the inverse $\psi^{-1} = \varphi_{-1}^W$ is obtained by going backwards in time. Choosing $W \in \mathcal{G}_3$ we get

$$\begin{aligned} \psi \circ F \circ \psi^{-1} \begin{pmatrix} x \\ y \end{pmatrix} &= DF(0) \begin{pmatrix} x \\ y \end{pmatrix} + F_2 + \ldots \\ &\quad + (X_W \circ DF(0)) \begin{pmatrix} x \\ y \end{pmatrix} - (DF(0) \circ X_W) \begin{pmatrix} x \\ y \end{pmatrix} + \ldots \end{aligned}$$

and simplifying the quadratic term of F amounts to solving the homological equation

$$(DF(0) \circ X_W \;-\; X_W \circ DF(0)) \begin{pmatrix} x \\ y \end{pmatrix} \;+\; B_2 \;=\; F_2$$

in the unknowns W and $B_2 \in \mathcal{G}_2 \times \mathcal{G}_2$. The linear operator

$$\begin{array}{rcccc} M^2_{DF(0)} & : & \mathcal{G}_2 \times \mathcal{G}_2 & \longrightarrow & \mathcal{G}_2 \times \mathcal{G}_2 \\ & & V_k(x,y) & \mapsto & (DF(0) \circ V_k - V_k \circ DF(0))\,(x,y) \end{array}$$

defines a splitting

$$\operatorname{im} M^2_{DF(0)} \;\oplus\; (\ker M^2_{DF(0)})^{\mathrm{T}} \;=\; \mathcal{G}_2 \times \mathcal{G}_2$$

and we are interested in the projections of F_2 to both factors. Inductively, once $F_2, \ldots, F_{k-1}$ have been normalized we similarly look for $V_k, B_k \in \mathcal{G}_k \times \mathcal{G}_k$ solving the homological equation for F_k. To compute the eigenvalues of $M^k_{DF(0)}$ we pass to complex co-ordinates $z = x + \mathrm{i}y$ and get

$$M^k_{DF(0)}(z^j \bar{z}^l) \;=\; \mathrm{e}^{\mathrm{i}\alpha} z^j \bar{z}^l \;-\; (\mathrm{e}^{\mathrm{i}\alpha} z)^j (\mathrm{e}^{-\mathrm{i}\alpha} \bar{z})^l \;=\; \mathrm{e}^{\mathrm{i}\alpha}(1 - \mathrm{e}^{\mathrm{i}(j-l-1)\alpha}) z^j \bar{z}^l$$

whence $M^k_{DF(0)}$ is semi-simple and the splittings

$$\operatorname{im} M^k_{DF(0)} \;\oplus\; \ker M^k_{DF(0)} \;=\; \mathcal{G}_k \times \mathcal{G}_k$$

hold true without the transpose. The kernel $\ker M^k_{DF(0)}$ consists of those monomials $z^j \bar{z}^l$ for which $j+l = k$ and $(j-l-1)\alpha \in 2\pi\mathbb{Z}$. For uneven $k = 2l+1$ this always yields $(z\bar{z})^l z \in \ker M^k_{DF(0)}$. In the resonant case that $\mathrm{e}^{\mathrm{i}\alpha}$ is an mth root of unity there are additionally $\bar{z}^{m-1} \in \ker M^{m-1}_{DF(0)}$, $z^{m+1} \in \ker M^{m+1}_{DF(0)}$, $\bar{z}^{2m-1} \in \ker M^{2m-1}_{DF(0)}$ and further combinations.

Exercise 8.4. Compute for

$$F(x,y) \;=\; \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x + y^2 \\ y \end{pmatrix}$$

the normal form of the quadratic term.

The difference in dimension

$$\dim \left\{ X_W \;\middle|\; W \in \mathcal{G}_{k+1} \right\} \;=\; k+2 \;<\; 2k+2 \;=\; \dim\left(\mathcal{G}_k \times \mathcal{G}_k\right)$$

corresponds to the k linear conditions on F_k for F to preserve area. In the simplest case $k = 2$ the quadratic terms

$$F_2 \;=\; \begin{pmatrix} ax^2 + bxy + cy^2 \\ dx^2 + exy + fy^2 \end{pmatrix}$$

have to satisfy

$$\det (DF_1 + DF_2) \equiv 1$$

whence

$$\begin{aligned} (2a+e)\cos\alpha + (2d-b)\sin\alpha &= 0 \\ (b+2f)\cos\alpha + (e-2c)\sin\alpha &= 0 \ . \end{aligned}$$

Note that a non-zero $\det DF_2$ has to be accounted for at the next level, where the coefficients of x^2, y^2 and xy are collected (making the 3 linear equations inhomogeneous). For the normal form series one has the following result of Takens.

Theorem 8.5. Let $F_\mu : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a parameter-dependent area preserving C^∞–diffeomorphism that has the origin as an elliptic fixed point for all parameter values. Then there is a family $H_\mu \in C^\infty(\mathbb{R}^2)$ of Hamiltonians and a family ψ_μ of area preserving diffeomorphisms such that

$$\left(\psi_\mu \circ F_\mu \circ \psi_\mu^{-1}\right)(x,y) = \left(\varphi_{t=1}^{H_\mu} \circ DF_0(0)\right)(x,y) + R(x,y,\mu)$$

with a remainder term R for which all derivatives vanish.

Thus, up to a flat remainder term (which is smaller than any polynomial close to the elliptic origin) the area preserving mappings F_μ are integrable.

8.3 The period-doubling bifurcation

The two exceptional cases $\operatorname{trace} DF(0) = \pm 2$ of double eigenvalues that are both on the real line and on the unit circle lead to bifurcations. The implicit mapping theorem does not apply to fixed points with eigenvalue $+1$ whence a small perturbation may remove such a fixed point. The logarithm $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ suggests to approximate F by the time–1–mapping of the Hamiltonian on top of p.10.

Exercise 8.5. Use the literature to show that a generic family of area preserving mappings encountering a parabolic fixed point with eigenvalue $+1$ undergoes a centre-saddle bifurcation.

The matrix $\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)$ does not have a real logarithm, but its square has the logarithm $\left(\begin{smallmatrix}0&-2\\0&0\end{smallmatrix}\right)$. In adapted co-ordinates the Hamiltonian H of a time–1–mapping approximating F^2 should be invariant under the $\mathbb{Z}_2$–action generated by $(x,y) \mapsto (-x,-y)$. This holds true for the families

$$H_\lambda(x,y) = -y^2 \pm x^4 + \lambda x^2 \tag{8.5}$$

which display Hamiltonian pitchfork bifurcations.

Exercise 8.6. Use the invariants $u = x^2$, $v = y^2$ and $w = xy$ to reduce the $\mathbb{Z}_2$–symmetry of the two families of Hamiltonian systems defined by (8.5) and draw in both cases phase portraits for a significant choice of values of the parameter λ.

Fixed points of F_λ^2 correspond for F_λ both to fixed points and to periodic orbits of period 2. In the above adapted co-ordinates the origin consists of fixed points while equilibria $(x, y) \neq 0$ of X_{H_λ} together with their symmetric counterpart $(-x, -y)$ correspond to a 2–periodic orbit of F_λ. Thus, under variation of a parameter λ a parabolic fixed point, with eigenvalues passing through -1, turns from elliptic to hyperbolic and simultaneously an orbit of period 2 bifurcates off from the parabolic fixed point. Depending on the sign ± 1 of the x^4–term the new orbit is elliptic and exists for parameter values λ for which the fixed point has become hyperbolic, or the new orbit is hyperbolic and coexists with the elliptic fixed points. One speaks of a period-doubling bifurcation, of supercritical and of subcritical type, respectively.